Numbers as Cognitive Tools

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Do numbers exist?

Most of the answers to this question presented in the literature of the last decades have relied on a priori methods of investigation, where scientific data and theories about the human experience of numbers are irrelevant.These a priori approaches, however, have been inconclusive.

In this dissertation, I adopt an empirically informed approach in which scientific descriptions of the human experience of numbers—as provided by cognitive sciences, linguistics, developmental psychology, and mathematics education—provide valuable information on the existence and status of what we call “numbers.”

These scientific descriptions allow for the conclusion that numbers, conceived of as independent, non-spatiotemporal objects, do not exist. What exist are certain human-made techniques which engender in us the idea that a special class of objects called numbers exists.

I show that, just as counting procedures and other arithmetical operations are cognitive tools that allow us to go beyond the limits of our genetically endowed cognitive skills, the very idea that numbers exist as independent objects is a cognitive tool that facilitates calculation—in other words, a useful reification.

The ontological hypothesis suggested by the scientific description of the human experience of numbers is that operations such as counting and calculating procedures are the objective subject matter underlying arithmetic, rather than a putative class of non-spatiotemporal objects.

Thus, the claim is that applied and pure arithmetical statements are true of the counting procedure and other arithmetical operations, rather than true of non-spatiotemporal numbers.

In contrast to other attempted answers to the question of the ontological status of numbers, the hypothesis defended in this dissertation is accountable towards empirical data, and can thus be improved or refuted on an empirical basis.

NUMBERS AS COGNITIVE TOOLS CÉSAR FREDERICO DOS SANTOS

NUMBERS AS COGNITIVE TOOLS

AN EMPIRICALLY INFORMED NOMINALISTIC ACCOUNT OF THE NATURE OF NUMBERS

CÉSAR FREDERICO DOS SANTOS

Numbers as Cognitive Tools

An empirically informed
nominalistic account of the
nature of numbers

César Frederico dos Santos

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VRIJE UNIVERSITEIT

Numbers as Cognitive Tools

An empirically informed nominalistic account
of the nature of numbers

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We can't dictate a priori the ontology of the universe.

Jody Azzouni (2015, p. 1148)

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Acknowledgments

It has been 1158 days since I started the PhD project that resulted in this dissertation. Not that I was counting the days like a prisoner, but it is a relief to see what once was a project converted into 233 pages that materialize in words the ideas that were revolving in my mind during each and every one of these days. Most of this work was done in solitude—only my laptop and I, for hours, in silence in my room (philosophy is not like field or lab research, even when it is empirically informed!). But others' contributions to this work were invaluable.

This work wouldn't have been possible without Catarina Dutilh Novaes. I found in her work the methodological approach to philosophy that I was looking for, and since we first met (in a café in Amsterdam in 2016) I have found in her the best supervisor I could ever wish for. Another essential person in this enterprise was Lieven Decock, my second supervisor. The best recommendations of philosophical literature that changed the face of this dissertation are his. I'm grateful to both my supervisors for their attentive reading and insightful discussion of my texts. Every time I met with them to discuss my drafts I came home with my mind bubbling with ideas.

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Introduction

THIS dissertation addresses a traditional philosophical question by means of a somewhat unconventional method. The question is: do numbers exist? To make this question tractable within the length of a dissertation, I restrict the investigation to the finite cardinal numbers (1, 2, 3, 4, …). The method I employ does not have a single name or characterization, but its basic tenet is the use of scientific findings to inform the investigation of philosophical problems. Methods of philosophical inquiry that comply with this tenet have been called naturalism (Quine, 1969), second philosophy (Maddy, 2007), experimental philosophy (e.g., Machery, 2017), or, more broadly, empirically informed philosophy (e.g., Dutilh Novaes, 2012), and, more recently, synthetic philosophy (Schliesser, 2019). These methods have been gaining traction in philosophy in the last decades, but philosophers of mathematics have been resistant to adopting them, particularly when it comes to ontological questions (with a few honorable exceptions, such as Maddy (1990)). The reason for this lack of enthusiasm for empirically-informed methods is straightforward: it is widely assumed that, if mathematical objects exist, they must be outside of space and time,[1] and therefore empirical data would be irrelevant to an investigation of the existence of mathematical entities.

The dominance of this view can be traced back to Frege (1960), the founding father of contemporary philosophy of mathematics, who advanced compelling arguments against numbers being either concrete or mental entities. If numbers are neither physical nor mental, the only remaining option seems to be to "place" them outside of space and time, if they exist at all. This move, however, drives the philosophy of arithmetic into a corner. If one assumes that numbers exist, their non-spatiotemporality leads to Benacerraf's problem (Benacerraf, 1983a). Outside of space and time, numbers are completely out of our reach, and it becomes a mystery how we can acquire numerical knowledge. On the other hand, if one rejects the existence of numbers, this poses problems of its own. For example, in fictionalism, the currently trendy version of nominalism, arithmetical statements are rendered false, since they supposedly speak of nonexistent things (Balaguer, 2018). For those who believe that there is something that can be properly called arithmetical knowledge, and reject platonism[2] exactly because it faces serious difficulties in explaining the origin of arithmeti-

[1]To give two literal examples: "if there are mathematical objects, then they exist outside of space and time" (Balaguer, 1998, p. 155); "If numbers exist, they would surely be abstract" (Giaquinto, 2001, p. 5).

[2]Platonism refers to the view according to which abstract entities (objects, structures, propositions, etc.) exist outside of space and time. Because it is contentious whether Plato himself endorsed this view, and because contemporary platonists are unlikely to endorse all of Plato's views on this matter, the term 'platonism,' when referring to this view in philosophy of mathematics, is spelled with lower-case initial (Balaguer, 2016).

cal knowledge, fictionalism is an equally unsatisfactory alternative. Other nominalistic approaches that reformulate or reinterpret arithmetical statements (such as Hellman's (1989)) usually involve artificial idealizations that, although logically impeccable, are practically unfeasible or unrealistic.

My response to this state of affairs is a typical philosophical move: if an assumption is impeding progress, put it aside and try to figure out an alternative. In this spirit, the starting point of this dissertation is the adoption of a neutral stance with regard to the existence and nature of numbers. An immediate consequence of this step back is that, once we no longer take for granted that, if numbers exist, they must be outside of space and time, we can start appreciating how empirical data could be relevant to an investigation of the ontological status of numbers.

In my view, the fundamental phenomenon to be explained in an investigation of the ontological status of numbers is the collection of relevant human experiences. Humans do arithmetic, speak of numbers, develop notation systems to refer to them, and use these things to help solve practical and theoretical problems. All of these practices take place in this world, and therefore are subject to scientific inquiry. What we are looking for in an ontological investigation of arithmetic is a class of entities that could account for the main features of these human experiences. Thus, instead of trying to figure out a priori what these entities are, we should instead rely on a scientific description of these experiences. Hopefully, this scientific description will reveal valuable information about which entities underlie our arithmetical experience—at least the spatiotemporal ones. Then, if we establish that spatiotemporal entities are insufficient to account for the main features of our arithmetical experience, this by itself will constitute evidence for the postulation of non-spatiotemporal entities. However, this is not a conclusion that should be established a priori.

Fortunately, there is plenty of scientific data available on the human experience of numbers. Numerical cognition is a branch of the cognitive sciences that investigates how human beings and other animals deal with information we usually see as numerical. Numerical cognition encompasses a number of disciplines, ranging from neurobiology and developmental psychology to linguistics and computer science. This field of investigation has progressed considerably in the last decades, and now it provides an illuminating picture of the roots of our numerical concepts and practices. Another source of data relevant to this enterprise is mathematics education. Educators have amassed considerable information about how we really come to acquire arithmetical knowledge at a young age.

I thus adopt a bottom-up approach: we look carefully at the scientific data gathered in these disciplines, look at the philosophical discussions of the ontological status of numbers, and try to extract from the data answers to the relevant philosophical question.

A first important finding for this investigation is the fact that human infants, long before they start learning to speak, possess the capacity to identify the cardinal size of small collections, up to three or four elements, quickly and accurately at a glance without counting—this ability is called subitizing. Above three or four elements, infants cannot do so, but they still can estimate cardinal sizes approximately, with estimate error increasing as the size of target collections grows. Many studies have repeatedly confirmed that the abilities to subitize the size of small collections accurately and to estimate the size of larger ones with increasing error are innate in humans and many non-human animals, such as monkeys and birds (Nieder, 2019). The abilities to subitize and to estimate are believed to constitute the most basic cognitive foundations of numerical competence in human beings (Dehaene, 2011). However, these innate abilities, although remarkable, are very limited in comparison to truly numerical competence, which is acquired only after years of training (Carey, 2009). To mention just one obvious example of how numerical competence outperforms these innate abilities, by counting we can determine the exact size of any collection, no matter its size (if sufficient time is provided), whereas by estimating we can hardly distinguish a collection of eight items from a collection of nine items. A fundamental difference between counting and subitizing or estimation is that the former is a rule-based procedure that makes use of a culturally created symbolic system—the list of counting words—whereas the latter are non-procedural and non-symbolic skills (after all, neither infants nor non-human animals master a symbolic system similar to our counting words upon which they could rely for subitizing or estimation).

The contrast between our limited innate non-symbolic abilities to deal with discrete quantities and the infinite potential of culturally created symbolic counting and calculation techniques suggests that the latter are a cultural response to the shortcomings[3] of the former (Barner, 2017). While genetic evolution endowed us, by means of subitizing, with the notion of exact quantities, it denied us access to those exact quantities beyond three or four elements. By counting, we become able to overcome this limitation. This suggests that counting is a cognitive technology specially developed by our ancestors to tackle the problem of assessing the cardinality of larger collections accurately. Likewise, other symbolic systems, such as the system of Arabic digits and the associated algorithms for arithmetical operations, can be seen as cognitive technologies created to facilitate calculation (Ifrah, 2000).

What these data suggest is that what we call "arithmetical knowledge" is knowledge of these cognitive technologies, of *techniques*. This runs against the traditional view according to which arithmetic is about a class of objects known as numbers. Certainly, we speak of numbers as if they were existing objects, and speak of arithmetic as if it were about numbers, but these are reified ways of speaking that, in fact, are about counting and calculating procedures. The idea that numbers exist as objects, as we will see, results from the encapsulation—a process of reification—of procedures into discursive objects, and fulfills the cognitive function of making arithmetical operations easier (Sfard, 2008). In this sense, the *idea* that numbers are objects is a cognitive tool in itself.

The scientific description of human experiences with numbers to be discussed here reverses the traditional platonist schema according to which first there were eternal numbers, then we somehow obtained representations of them, and then these representations finally allowed us to count, calculate, and do arithmetic in general. In the sequence of events that takes place when a typical child is learning arithmetic, and which most likely took place in remote history, first the child learns techniques for counting and calculating; from these techniques, she acquires the idea of an ordered sequence of cardinal values; in a later stage, she starts speaking and thinking of this sequence as constituted by a certain kind of object called numbers (Carey, 2009; Sfard, 2008). In the bottom-up approach adopted here, the

[3]That is, "shortcomings" in view of the need to determine the exact cardinal size of collections with more than three or four items.

ontological hypothesis suggested by this reversal of the platonist schema is that numbers as such do not exist. What exists are certain human-made techniques which engender in us the idea that a special class of objects called numbers exist. In other words, the suggestion is that operations such as the counting procedure are the objective subject matter underlying arithmetic, rather than a putative class of non-spatiotemporal objects. The hypothesis to be defended here is that arithmetical statements are true of these procedures, rather than true of non-spatiotemporal numbers.

The initial idea for this project came from reading Dutilh Novaes's *Formal Languages in Logic* (2012) and Dehaene's *Number Sense* (2011) in parallel. I observed that Dutilh Novaes's account of the role of formal languages in deductive reasoning could be almost directly translated to the role of numerals in numerical cognition. Dutilh Novaes shows that formalisms are cognitive technologies which enable us to overcome cognitive tendencies that make deductive inferences considerably hard for non-trained minds. Dehaene, in turn, suggests that numerals have the power to sharpen our innate ability to estimate cardinal sizes only approximately to the point of converting it into the ability to count and understand exact cardinal sizes. The analogy is clear: *numerals* are like formalisms in that both are part of humanly invented techniques that boost our cognitive abilities.

The idea that "numbers" are cognitive technologies is not new. In non-philosophical circles, this idea has been around for a while. The linguist Heike Wiese holds that "numbers are flexible mental tools that can be employed in a wide range of contexts" (Wiese, 2003, p. 292). In what he calls "a street-level, economic ontology of numbers," the economist David Harper, inspired by Wiese (2003), claims that "numbers are technological objects ('tools' in the widest sense of the term) that are constituted by both form and function" (Harper, 2010, p. 170). The anthropological and cognitive linguist Caleb Everett holds that "numbers were and remain cognitive tools—tools that transformed our lives long before the usage of advanced mathematics" (C. Everett, 2017, p. 258). In the field of numerical cognition, perhaps the most notable occurrence of this idea is in the title of the highly influential paper by Frank, Everett, Fedorenko, and Gibson (2008) titled "Number as a cognitive technology: Evidence from Pirahã language and cognition." Based on their investigation of Pirahã, a language spoken by a small group of Amerindians in the Brazilian Amazon which is the only known language without numerals, they conclude that "numbers may be better thought of as an invention: A cognitive technology for representing, storing, and manipulating the exact cardinalities of sets" (Frank et al., 2008, p. 823).

This idea has made its way into philosophy too. However, philosophers, who are well aware of the necessary distinction between *numerals* and *numbers*, are more reluctant to claim that *numbers* are cognitive tools. Without an account of how numbers, widely held as non-spatiotemporal entities, could be a human invention, philosophers feel more comfortable saying that *numerals* are cognitive technologies. Speaking of mathematical symbols in general, De Cruz and De Smedt (2013, p. 17) write: "[u]sing mathematical symbols can be seen as epistemic actions, not unlike the use of other external tools in scientific practice, such as microscopes, particle accelerators and slide rulers." In a similar spirit, Menary (2015, p. 14) claims that "[m]athematics and writing systems are examples of culturally evolved symbol systems that are deployed to complete complex cognitive tasks." In the literature on 4E cognition, mathematical symbols are often mentioned as examples of cognitive tools (e.g., Clark, 2008). The novelty of my proposal is that I offer a philosophical account of how *numbers* themselves—not the putative non-spatiotemporal entities, but the discursive objects numerals refer to, in a sense to be specified in Chapters 6 and 7—can be conceived of as cognitive tools in their own right.

Synopsis of the dissertation

This dissertation follows a threefold structure. The first part comprises Chapters 1 and 2. In these chapters, I motivate the empirically informed methodology adopted. The second part comprises Chapters 3 to 6, where I review the scientific data and theories relevant for the present investigation. The third part comprises the final chapter (Chapter 7), where I return to the question of the ontological status of numbers, now informed by the scientific theories discussed. In the following paragraphs I summarize the main points discussed in each chapter.

In Chapter 1, I show that the aprioristic methods that have pervaded the philosophy of mathematics cannot offer a conclusive answer to the question of the ontological status of numbers. I argue that at the heart of this impossibility lies a tension between the idea that the existence (or nonexistence) of numbers is an objective fact—either they exist or they do not, and this is not up to philosophers to decide—and the related need to provide independent evidence for the ontological claims made by a priori accounts. A priori methods cannot provide this kind of evidence and, therefore, there is no conceivable a priori means of verifying the ontological claims made by a priori accounts. I argue that such ontological claims should be put aside as unverified or unverifiable hypotheses. The inconclusiveness of a priori methods suggests that we should broaden the scope of the investigation and take into account scientific data about the human experience of numbers.

In Chapter 2, I propose an empirically informed methodological approach to the question of the ontological status of numbers. In this approach, the most fundamental question concerns how human beings acquire numerical competence. One of the most important findings in numerical cognition is that numerical competence is inseparable from the use of symbolic systems for numbers. The fact that we cannot count and calculate accurately without mastering a symbolic system for numbers brings to the fore the instrumental role of numerals as tools which enhance and expand our cognitive capacities, and relates to the broader philosophical discussion about symbolic cognitive tools. In Chapter 2 I review central topics in the literature on cognitive tools that are key to make sense of the scientific data about numerical cognition we will see in this and subsequent chapters. In the final section of Chapter 2, I sketch the cognitive, epistemic and ontological hypotheses about numbers that I will further investigate in the following chapters.

In Chapter 3, I discuss the innate abilities to deal with discrete quantities that humans share with non-human animals. I review evidence for the existence of these abilities, characterize them precisely, and investigate whether they can be properly qualified as numerical, as many cognitive scientists have claimed. Based on a critical assessment of the terminology scientists use to describe and explain these abilities, I conclude that they are non-numerical and, therefore, are better called *quantical* abilities, following Núñez (2017). This has an important consequence: since number concepts are not a product of genetic evolution, everything we know about numbers, from counting to formal arithmetic, must be a product of cultural processes.

In Chapter 4, I review further scientific findings that reinforce the hypothesis of the cultural origins of numerical concepts. The focus of Chapter 4 is the ontogenetic development of numerical cognition, i.e., the processes through which children acquire numerical competence. If numbers are a cultural creation, we must be able to observe children learning number concepts from their parents or caregivers through regular processes of enculturation, as is usually the case with culturally transmitted contents. As we will see, this is exactly what studies in developmental psychology show. These studies also highlight the importance of symbolic systems for numbers for the acquisition of numerical competence, and particularly the role of the counting procedure in giving rise to our concept of number.

In Chapter 5, the focus is on the historical origins of number concepts. Since, as shown in Chapter 4, number concepts originate from symbolic techniques passed down across generations through cultural transmission, there must have been a time when people without number concepts invented the symbolic techniques that gave rise to them for the first time in history. Linguistic and ethnographic evidence reviewed in Chapter 5 suggest that counting was created from earlier non-numerical one-to-one correspondence techniques.

All things considered, the conclusion that emerges from these chapters is that numerical competence results from the acquisition of human-made symbolic techniques. That is, we acquire number concepts not by obtaining information about a certain class of objects, but by mastering a toolkit of techniques. The piece of the puzzle still missing is how we move from these techniques, which engender in us number concepts, to the idea that these concepts are about existing entities, namely numbers. I address this question in Chapter 6. We will see that this transition takes place through a process of reification in which we start speaking of the counting procedure and its outcomes by means of the linguistic framework we use to talk about ordinary objects. Chapter 6 explores Sfard's (2008) theory of reification in the field of mathematics education to explain how this happens. In the bottom-up approach adopted here, the observation that the idea that numbers exist originates from a process of reification suggests *nominalism*: numbers do not exist, although we speak of them as if they do.

In order to make this hypothesis philosophically plausible, in Chapter 7 I provide a nominalistic account of the semantics and epistemology of arithmetic in which counting procedures play the role traditionally ascribed to numbers as existing objects in ensuring the epistemic attributes of arithmetic, such as truth and objectivity. In this account, instead of describing the properties of a non-spatiotemporal realm of existing numbers, arithmetic describes structural properties of the cognitive tools that constitute it.

In the final pages of this dissertation I provide a glossary where the reader can find definitions of some concepts and less familiar terms used in the work.

Chapter 1

A methodological shortcoming in the philosophical literature on numbers

The recent decades have seen many tentative answers to the question of the existence of numbers. Most of them rely chiefly on aprioristic methods. By 'aprioristic methods' I mean those methods of investigation that neither produce nor rely on empirical data. Some examples are conceptual analyses that rely mostly on linguistic intuitions, instead of using data about language usage gathered scientifically; philosophical considerations justified by appeal to theoretical virtues such as simplicity and uniformity; and appeal to logical, mathematical, or semantic principles and results. The prevalence of aprioristic approaches to the question of the ontological status of numbers is perhaps an influence of the view that "if numbers exist, they exist outside of space and time"—case in which empirical data is irrelevant to addressing this question.

Contrary to this view, in this chapter I show that aprioristic methods are not suited to establish whether numbers exist in a conclusive way. The most we can do with aprioristic methods, I argue, is to advance *hypotheses* about the existence (or nonexistence) of numbers. However, these hypotheses can never be conclusively confirmed or rejected by purely a priori arguments. The reason is simple and it is concisely expressed in the Azzouni quotation which is the epigraph to this dissertation: "we cannot dictate a priori the ontology of the universe" (Azzouni, 2015, p. 1148). A priori considerations are introspective; by definition, they do not take into account worldly facts, whereas whether or not numbers exist is a worldly fact (even if it is not a fact about this world, but about a non-spatiotemporal reality). By a priori reflection, therefore, we can only *speculate* about how the world *might* be. Without access to the relevant facts we cannot decide whether our preferred speculative account of the world corresponds to the way the world is.[1]

It may be that the existence of numbers is dependent on human minds, or social institutions; or it may be that numbers exist independently, in a platonic realm; or, rather, it

[1]This limitation does not affect only the investigation into the ontological status of mathematical entities but other metaphysical investigations too. Nolan (2016, p. 176) writes: "Some contemporary metaphysicians (myself included) find it strange that it should ever have seemed plausible that metaphysics was an a priori discipline. Why suppose we could tell general and deep facts about what the world as a whole is like without evidence from our senses?"

may be that they exist in a completely different manner, or even that they do not exist at all. Either way, there is an objective fact to be discovered by the philosopher. The difficulty the philosopher who relies exclusively on aprioristic methods has to face, then, is that whenever one gives an account of what is supposed to be an objective reality, she is required to provide evidence that her account is *correct* about that reality, or at least show how such evidence could be obtained. Importantly, this evidence should be independent, at least to a certain degree, of the given account, on pain of circularity. As we will see, aprioristic methods cannot provide such independent evidence. Equipped only with aprioristic methods, we are necessarily confined to our human, internal, rational faculties. *Pace* Descartes, from within ourselves we cannot dictate, nor discover, what exists out there.

For certain, philosophers are well aware of this difficulty, and have advanced many strategies to deal with it. Because different strategies fail for different reasons, I do not have a general argument to show that, in every case, aprioristic methods will not do. For this reason, in this chapter I proceed case by case, and consider a limited but representative sample of philosophical accounts of the ontology of mathematics proposed in the last decades. On the platonist side, I examine Hale and Wright's neo-Fregeanism (Hale & Wright, 2001), Shapiro's *ante rem* structuralism (Shapiro, 1997) and Resnik's structuralism (Resnik, 1997). On the anti-platonist side, I consider Hellman's modal-structuralist nominalism(Hellman, 1989), Bueno's two forms of fictionalism (Bueno, 2009), and Leng's fictionalism (Leng, 2010). I also consider two deflationist approaches that deny that there is a deep philosophical problem to be solved about the ontology of mathematics: Thomasson's simple realism (Thomasson, 2015) and Maddy's thin realism (Maddy, 2011). I do not have knock-down arguments against these philosophical approaches, but I will show that they are inconclusive. The inconclusiveness of each approach taken in isolation, and, consequently, the seemingly inconclusiveness of the a priori program as a whole is perhaps the best reason we have to suspect that a priori methods are not suited to answer the question of the existence of numbers. In the following sections I discuss each of these accounts in turn.

1.1 Neo-Fregean platonism

Crispin Wright and Bob Hale are the leading figures in neo-Fregeanism. Their argument for the existence of numbers is straightforward. Its central premise is that the occurrence of singular terms in true statements implies the existence of their referents. For example, from the truth of 'Mars has two moons,' we can safely infer that Mars exists, since Mars is the referent of the singular term 'Mars.' Now, insofar as there are true numerical statements where numerals occur as singular terms, we can safely infer (or so they argue) that numbers exist.

> Provided, then (as certainly appears to be the case), there are true extensional statements so featuring numerical singular terms, there are objects—numbers—to which they make reference. This train of thought constitutes the core of the style of platonism we find in Frege—at any rate, the style of platonism we wish to defend (Hale & Wright, 2001, p. 8).

It is important to note that, as they see it, numbers enjoy full existence, i.e., they are "furnishings of the world every bit as objective as mountains, rivers and trees" (Wright, 1983,

p. 13). Mountains, rivers and trees exist independently of our linguistic expressions. The existence of trees on the top of a mountain does not depend on someone naming them, or referring to them in any sense. We can confirm this by climbing the mountain and seeing that there are trees there. This is the kind of independent existence they claim for numbers. However, their argument for the existence of numbers does not rely on a direct inspection of the "top of the mountain," so to speak, but on an analysis of our *discourse* about "the trees on the top of the mountain," so to speak.

Indeed, Hale and Wright acknowledge that "th[eir] argument must succeed unless *either* the apparent singular terms of arithmetic do not really function as such *or* the apparently true 'appropriate' contexts in which they feature are not really true" (Hale & Wright, 2001, p. 154). Both alternatives have found advocates. Hofweber (2005), for example, claims that number words are not genuine singular terms, but determiners that, occasionally, can occur syntactically as singular terms. Fictionalists, as we will see below, accept that numerals are singular terms, but deny that numerical statements are true. Furthermore, contrary to what the above quote suggests, these are not the only ways in which the neo-Fregeans' argument can be wrong. As we will see below, it may be that numerals are genuine singular terms but that they refer to positions in a structure, rather than to objects, as Shapiro (1997) argues, or that they refer to possible, rather than actual, objects, as Hellman (1989) argues. In face of so many alternatives, how can we figure out which analysis of numerical statements is the correct one?

The usual a priori strategy to compare competing accounts is to focus on their conceptual and logical details. The inference of an unwelcome consequence, the existence of counterexamples, the presence of theoretical shortcomings; these can be signals that the proposed analysis of numerical statements is defective. A quick look at the philosophical debate, however, suffices to show that this strategy has been inconclusive. Advocates of a particular "failed" proposal do not abandon it, but modify it in a way that avoids the undesirable consequence and preserves its central ontological claims. Neo-Fregeanism is a well-known example of the rehabilitation of a program that was thought to be irremediably defeated by the presence of a contradiction. The contradiction was eliminated and the ontological status of numbers implied by the original program remained unaltered.

A more conclusive way of adjudicating between competing accounts could follow the scientific practice of crucial tests: if there are theories A and B such that A predicts that x exists and B predicts that x does not, an experiment designed to detect the existence of x can be crucial to decide which is correct. Platonist accounts of arithmetic, however, cannot be put to such a test, since their advocates carefully add that numbers are outside of space and time and, therefore, inaccessible to any experimental approach *by definition*. I emphasize 'by definition' because the non-spatiotemporality of numbers is not implied by platonists' analyses of numerical statements. For example, the neo-Fregean argument for the existence of numbers only implies that numbers exist, but says nothing about their mode of existence. Their non-spatiotemporality is an additional postulation, apparently aimed at making the existence of numbers more plausible. "Since it appears to make no sense to ask where numbers or sets are located, or when they came into existence and how long they will last, the platonist concludes that they are abstract objects, lying outside space and time" (Hale & Wright, 2001, p. 169).

The transference of numbers to an inaccessible, non-spatiotemporal realm can be seen as an *ad hoc* way of dismissing requests for *independent* evidence for their existence. Neo-Fregeans, though, are ready to respond to this objection. Wright argues that demands for independent evidence for the existence of numbers are not only dispensable, but also illegitimate: they are forbidden by Frege's Context Principle. In Wright's reading of it, the Context Principle states that we should never ask after the *referent* of a term in isolation, only in the context of a proposition. "To suppose that such a question [whether a numerical expression denotes any genuine constituent of the world] does arise is exactly to suppose that it is legitimate to inquire whether such an expression genuinely denotes anything in isolation from consideration of the part which it standardly plays in whole propositions" (Wright, 1983, p. 14). Since no such question can be posed without violating the Context Principle (or so he argues), Wright derives from it an important ontological corollary:

> the thesis of the priority of syntactic over ontological categories. According to this thesis, the question whether a particular expression is a candidate to refer to an object is entirely a matter of the sort of syntactic role which it plays in whole sentences. If it plays that sort of role, then the truth of appropriate sentences in which it so features will be sufficient to confer on it an objectual reference (Wright, 1983, p. 51-52).

This has become known as the *syntactic priority thesis*. Wright invokes this thesis with the intention of lending further support to his conclusion about the existence of numbers.

> Given the relevant facts about syntactic structure and truth, there is simply no further coherent question about the matter; there is no independent philosophical issue whether there really is such a thing as the number 7, still less any possibility of an independent resolution of such an issue (Wright, 1983, p. 14).

The idea seems to be that the ban on requests for independent evidence for the existence of numbers can provide support for his account. However, the syntactic priority thesis, if true, is neutral about which is the correct account of numerical discourse. Different accounts of numerical discourse may identify different syntactic categories in numerical expressions. All of them will be equally backed by the syntactic priority thesis with respect to their ontological implications. The relevant point, for which the syntactic priority thesis is of no help, is to figure out which is the correct syntactic account of numerical statements. Now, if the Context Principle really forbids any independent crucial test of competing accounts, this is also bad news: we cannot hope to solve this question by recruiting what is a familiar method in other fields. At any rate, Wright's overly strict interpretation of the Context Principle can be seen as a further *ad hoc* way of dismissing requests for independent evidence for neo-Fregeans' existential claims.

Another way of bypassing requests for independent evidence is to claim that the existence of numbers is an analytic truth. If this is the case, as neo-Fregeans claim, then no further evidence would be needed. According to them, the existence of numbers can be proven in second-order logic from Hume's Principle, which might be analytic (Wright, 1999). Putting concerns about second order logic and the notion of analyticity aside, their derivation of numbers from Hume's Principle is flawless. However, it cannot be overlooked that Hume's Principle is not the only premise of this proof. Their own account of the syntax of numerical expressions also plays a central role in it. Let me further clarify this point.

Hume's Principle states that *the number of Fs is equal to the number of Gs iff F and G are equinumerous*. The first step to obtaining the existence of numbers from Hume's Principle, as Wright (1999, p. 8) explains, is to obtain the existence of zero. To obtain zero, let *F* and *G* be the set of *x*s such that $x \neq x$. It is a logical truth that thus given, *F* and *G* are equinumerous. Therefore, we have the right-hand side of Hume's Principle, and can immediately infer its left-hand side: the number of *F*s is equal to the number of *G*s. This, then, is an analytically proven statement. Now comes the decisive step: to infer the existence of zero from the left-hand side of this instance of Hume's Principle we must assume that the expression 'the number of *F*s' (or *G*s) is referential. Because Wright formalizes this expression in second order logic as a referring term, he obtains the existence of zero. But it is by no means an analytic truth that expressions of the form 'the number of *x*s' are referring expressions. They are in neo-Fregeans' analysis, but neo-Fregeans' analysis is precisely what is under scrutiny here. Therefore, even if Hume's Principle is analytic, the existence of numbers does not follow analytically from it.

We are not required to accept the ontological consequences of neo-Fregeanism as long as it remains undemonstrated, in a non-circular and non-ad-hoc way, that its account of numerical expressions is the correct one, and that the ontology its proponents derive from it really exists. Although neo-Fregeans are unable to provide independent evidence for these claims, nothing that was said above implies that neo-Fregeanism is wrong. Thus, let us take it as a hypothesis that makes unconfirmed ontological predictions, for which we cannot devise any a posteriori way of testing, nor any a priori way that does not beg the question.

1.2 Shapiro's *ante rem* structuralism

Some philosophers resort to structuralism in order to obtain nominalism. This is what Hellman does (see section 1.4). Shapiro (1997), by contrast, resorts to structuralism to obtain an amplified form of platonism. Shapiro's platonic heaven is more populated than Hale and Wright's. For Shapiro, the structuralist motto according to which "mathematics is concerned with the study of structures" does not imply that particular mathematical objects do not exist; rather, it implies that structures *and* particular objects exist. Let us examine Shapiro's account.

Similarly to neo-Fregeans' platonism, Shapiro's *ante rem* structuralism relies on a logical analysis of mathematical discourse carried out in second order logic. His analysis is quite similar to neo-Fregeans'. He agrees with neo-Fregeans that, insofar as singular terms that refer to numbers occur in numerical statements held to be true, numbers must exist. However, Shapiro points out the fact that nonalgebraic[2] theories formalized in second order logic have multiple isomorphic—but not identical—models. This is the case for arithmetic, and this makes the task of determining the identity of the referents of numerical terms quite difficult.

A well-known illustration of this point is presented in Benacerraf (1983b). As Benacerraf

[2]In Shapiro's terminology, a theory is algebraic if it is not intended to be about a single structure, unique up to isomorphism. Group theory is an example. For algebraic theories, there is no point in identifying *the* unique objects which the terms of the theory refer to. Nonalgebraic theories, by contrast, are intended to be about a single structure or isomorphism type (Shapiro, 1997, p. 40-41).

remarks, both Zermelo's and von Neumann's ordinals are isomorphic models of arithmetic. In both models, '0' refers to $\varnothing$ and '1' refers to $\{\varnothing\}$. However, from two onward, the models start to differ. In Zermelo's approach '2' refers to $\{\{\varnothing\}\}$, whereas in von Neumann's approach '2' refers to $\{\varnothing, \{\varnothing\}\}$. This means that in von Neumann's model, 0 belongs to 2, whereas in Zermelo's, it does not. Which one is the real 2? Because both models equally satisfy Peano's Axioms, we do not have any reason to prefer one over the other. This result suggests that it does not matter which particular object plays the role of a number, provided that the relations between this object and the objects that play the role of the other numbers are preserved. The conclusion that Benacerraf draws from this is that numbers are not objects at all. Shapiro, however, rejects Benacerraf's conclusion and tries to reconcile the suggestions that numbers are objects and that it does not matter which object plays the role of each particular number.

Roughly put, Shapiro's idea is that each model of a mathematical theory is an instance of the structure described by the theory. Structures are characterized by implicit definitions that take the form of axioms. In arithmetic, for example, Peano's Axioms characterize the natural-number structure. In Shapiro's view, there is only one, unique, natural-number structure, but there are infinitely many systems that instantiate it. Zermelo's and von Neumann's ordinals are two of these systems. For Shapiro, structures are like universals, whereas the systems that instantiate them are like particulars. "The difference between structures and the more usual kind of universal, such as properties, is that structures are the forms, not of individual objects, but of systems, collections of objects organized with certain relations" (Shapiro, 2008, p. 302). Moreover, structures and platonic universals share the characteristic of being *ante rem*, i.e., they exist prior to and independently of any items that may instantiate them (Shapiro, 1997, p. 84).

Thus, in Shapiro's platonic heaven there are both *ante rem* structures and the systems that instantiate them. Particular numbers still cannot be identified with objects of a certain system, but because the natural-number structure is *ante rem*, particular numbers can be identified with *places* in this structure. For Shapiro, places in structures are objects in their own right. He explains this point by making an analogy between places in a structure and offices in an organization. The office of the vice president, for example, exists even if it is vacant, and is distinguishable from the person who holds that office at a particular moment. "Similarly, we can distinguish an object that plays the role of 2 in an exemplification of the natural-number structure from the number itself. The number is the office, the place in the structure" (Shapiro, 1997, p. 77). In doing so, Shapiro can treat numerical expressions as singular terms: '0' denotes the first place in the natural-number structure, '1' denotes the second place, and so on.

Now we have two competing platonist accounts. In Shapiro's account, the platonic heaven contains *ante rem* structures; in the neo-Fregeans', it does not (or, if it does, this is not particularly relevant for an account of the ontology of arithmetic). Which one is true of the way things are in this heaven? Shapiro's argument for the existence of *ante rem* structures, just like neo-Fregeans' argument for the existence of numbers, does not rely on a direct inspection of the the platonic heaven (which is impossible to do, by definition) but on an analysis of mathematical discourse.

Despite the absence of direct evidence of this sort, Shapiro argues that there is no reason for concern: his analysis cannot be wrong about what exists in the platonic heaven, since his analysis is "coherent." Shapiro understands 'coherence,' roughly, as satisfiability (Shapiro, 1997, p. 133) and his idea is that the "ability to coherently discuss a structure is evidence that the structure exists" (Shapiro, 1997, p. 118). Let us call this the *coherence-implies-existence thesis*. As we have seen above, neo-Fregeans' syntactic priority thesis implies the existence of the referent of a singular term only if there is a true statement in which that singular term occurs. The coherence-implies-existence thesis is more liberal in this regard. For a structure to exist, it is not required that the axioms be true, only satisfiable.

The problem, again, is that an independent reality can always be uncooperative with regard to our theories about it. Thus we have to ask what evidence can support the claim that to every satisfiable axiomatization we devise in this world there corresponds a structure in the platonic heaven. Without independent evidence for this, the coherence-implies-existence thesis is no more than a hypothesis.

In response to this challenge, Shapiro recognizes that traditional platonism is subject to such risks, but denies that his coherence-implies-existence thesis would fail in this respect. The difference is that, "on the traditional Platonist view, there is an important autonomy between the axioms and the subject matter," whereas "[w]e structuralists reject this autonomy" (Shapiro, 1997, p. 131). It is worthwhile to go through his explanation of this point in full:

> Because on the traditional Platonist view, the axioms are statements about a particular realm of objects, it is possible that the axioms can be mistaken. Perhaps there are natural numbers other than zero that have no successors. Perhaps the successor function is not one-to-one. As a thought experiment, try to consider the skeptical possibility that all of the Peano axioms are false of the natural numbers. A traditional Platonist is faced with such a possibility. Because the referent of "the natural numbers" is somehow independent of the characterization in the language of arithmetic, any given belief and, indeed, every (nonlogical) belief we have about numbers might be false. For the structuralist, on the other hand, this extreme skeptical possibility can be dismissed out of hand. It is conceivable, barely, that arithmetic is incoherent, in which case no structure is characterized. Perhaps the theory of arithmetic is not categorical, in which case more than one structure is characterized. But it is nonsense to claim that the theory of arithmetic does successfully refer to a single, fixed structure (or a fixed class of structures) but says hardly anything true about it (or them). On our view, the language characterizes or determines a structure (or class of structures) if it characterizes anything at all (Shapiro, 1997, p. 131).

Shapiro's notion of 'characterization' is the key to understanding his claim that Peano's Axioms cannot be wrong about the natural-number structure. To characterize a structure, for Shapiro, is not the same as to describe it. If we understood Peano's Axioms as giving a description of the natural-number structure, then they could be wrong about it. To characterize a structure is not the same as to stipulate its properties either, Shapiro claims, because in this case the existence of the structure would be dependent on our stipulations. This would amount to a kind of conventionalism, but

> [s]tructuralism is not a general skepticism nor a conventionalism. Mathematics is objective if anything is. The natural-number structure has objective existence and facts

> about it are not of our making. The point is that the way humans apprehend structures and the way we 'divide' the mathematical universe into structures, systems, and objects depends on our linguistic resources. Through successful language use, we structure the objective subject matter. Thus, language provides our epistemic access to mathematical structures (Shapiro, 1997, p. 137).

Thus, to characterize a structure, in Shapiro's view, is to "divide" the mathematical universe using our linguistic resources. The claim seems to be that there is an objective subject matter, but the way we structure it depends on our linguistic resources. Insofar as the only requirement imposed by our linguistic resources is coherence, in Shapiro's view, coherence implies existence.

If this is so, though, it is hard to see how the putative objective subject matter plays any role in Shapiro's account. If the only constraint on the specification of mathematical structures comes from language, the way the mathematical universe is organized turns out to be irrelevant. Given that mathematicians are free to specify any structure that results from successful language use, it does not matter how things are in the putative independent mathematical universe. It is hard to see why Shapiro's account would not be conventionalist. Azzouni (2000, p. 232) has something to say about this:

> Some philosophers of mathematics marry an ontologically independent mathematical realm to a stipulationist epistemology. The result is unstable if only because such a union still craves explanation for why the stipulations in question correspond to the properties of the ontologically independent items they are stipulations about.

If conventionalism is to be avoided, Shapiro must provide an explanation of why coherent characterizations of mathematical structures carve the mathematical universe at its joints. However, it is not even clear whether Shapiro's platonic heaven has "joints." His allegation that "the way *we* 'divide' the mathematical universe into structures … depends on *our* linguistic resources" (as quoted above; emphases added) seems to suggest that the mathematical universe is originally amorphous and unstructured, and thus compatible with any coherent structure we impose on it. On the other hand, his affirmation that "[t]he natural-number structure has objective existence and facts about it are not of our making" (as quoted above) seems to suggest the opposite, i.e., that the mathematical universe has a structure that is independent of the way we divide it. These claims hardly seem compatible; a clear signal of the unstable marriage that Azzouni refers to.

In the absence of independent evidence for the existence of *ante rem* structures, then, the coherence-implies-existence thesis remains a puzzling metaphysical hypothesis.

1.3 Resnik's structuralism

Resnik, as Shapiro, combines platonism and structuralism. His structuralist account of mathematics is quite similar to Shapiro's. For Resnik, "[t]he objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are themselves atoms, structureless points, or positions in structures" (Resnik, 1997, p. 201). Thus, Resnik, as Shapiro, preserves the standard reading of mathematical statements by identifying the referents of mathematical terms with places in structures. Moreover, Resnik, as Shapiro, believes that structures and their positions exist independently of us in a platonic realm. He characterizes his realism in this way:

> My realism consists in three theses: (1) that mathematical objects exist independently of us and our constructions, (2) that much of contemporary mathematics is true, and (3) that mathematical truths obtain independently of our beliefs, theories, and proofs (Resnik, 1997, p. 4).

Resnik differs from Shapiro, however, in the argument he uses to support platonism. Resnik is admittedly a stipulationist, and he recognizes that combining stipulationism and realism is not an easy task. The basic idea of his approach

> is that humans brought mathematical objects into their ken by positing them. Now to posit a new kind of object one need only introduce a new predicate *P* (or, as happens frequently, begin to use an old one with a new sense) and claim that *P* exists. Thus, it is plain that realists who claim that mathematical objects are posits invite a variety of worries and objections. Postulational approaches seem better suited to conventionalists, who may claim that we make truths, than to realists, who must hold that we can only recognize independently obtaining truths (Resnik, 1997, p. 175).

As a realist, he admits that "it is essential to establish that a postulational epistemology is compatible with realism" (Resnik, 1997, p. 188), and he believes that this is feasible. His point is that we do not need direct access to the independent realm of mathematical objects in order to check whether our posits are really there. The justification for believing that, e.g., numbers, whose existence we postulate, *really exist* in the platonic realm can be established through other means that do not require any access to the numbers themselves; namely, through the indispensable use of numbers in science. Resnik endorses the Quine-Putnam Indispensability Argument (Putnam, 1975; Quine, 1963). Naturally, this rationale justifies only belief in those parts of mathematics that are applied in science. Resnik consistently follows Quine in this regard and also denies ontological rights to parts of pure mathematics.

Resnik's argument stands out from the other platonist positions I consider in this chapter because he attempts to provide the kind of independent evidence for the existence of mathematical entities I am asking for. The observation that arithmetic and other mathematical theories are indispensable to science is independent of any logical analysis of mathematical statements. However, this very independence should prevent us from assuming that the indispensability of mathematics provides evidence for a particular logical analysis of mathematical theories. Let me unpack this point.

For the sake of argument, let us concede that the indispensability thesis provides evidence for believing in the truth of mathematical theories. Can we infer from the truth of mathematical theories that mathematical entities exist? The answer is affirmative only if we assume the standard reading of mathematical statements, as Resnik does. However, given that the correct reading of mathematical statements is controversial and many reject the standard reading, this assumption is not trivial. One may hold that mathematics is true but deny that it makes existential claims by reinterpreting mathematical statements. This is what Hellman (1989) does, just to mention an account I consider in this chapter. Because evidence for the truth of mathematical theories does not carry over to philosophical interpretations of mathematical statements, the indispensability thesis by itself cannot be seen as providing evidence for the existence of mathematical entities. If correct, the indispensability thesis only establishes that mathematical theories applied in science are true. Although this can be seen as a first step towards platonism, it is not a conclusive step.

1.4 Hellman's nominalism

To avoid untestable hypotheses about a non-spatiotemporal world, perhaps we should embrace a form of nominalism. Many nominalist approaches adopt a common strategy: they reformulate mathematical discourse in a way that eliminates any reference to abstract objects. To achieve this, some of them make use of modal notions. This is the strategy adopted by Hellman (1989).

Hellman proposes a modal-structuralist *reconstruction* of mathematical language. In contrast to neo-Fregeans' and Shapiro's analyses of mathematical discourse, Hellman's reconstruction is not intended to reveal the true syntax of mathematical statements, but to provide an alternative way of reading them. He recognizes that "if one simply reads ordinary mathematical discourse literally, i.e. takes it 'at face value', one arrives at a platonist interpretation" (Hellman, 1989, p. 2). Once this is admitted, the first thing one has to do in order to make room for nominalism is to dismiss neo-Fregeans' syntactic priority thesis. To this end, Hellman relies on a kind of "bad-company argument." The idea is that we cannot assume that the truth of statements wherein a singular term occurs implies the existence of its referent because there are many apparently true statements about things that do not exist—the bad companies. Hellman's examples:

> [T]here are hordes of counterexamples [to the syntactic priority thesis]: "Thor was the Nordic god of thunder", "Odysseus charted a safe course between Scylla and Charybdis" etc. But how does one distinguish the relevant from irrelevant truths? Surely not on any straightforward syntactic grounds (Hellman, 2001, p. 695).

Even if the syntactic priority thesis is applied to abstraction principles only, there are still counterexamples. Hellman (2001, p. 695) mentions this: "The god of person P = The god of person Q iff P and Q are coreligionists," which could entail the existence of the god of persons P and Q. Hellman concludes: "I see no non-circular way of restricting the 'syntactic priority thesis' for it to have a prayer" (Hellman, 2001, p. 696).

The rejection of the syntactic priority thesis makes room for the claim that the occurrence of singular terms in true mathematical statements *does not* entail the existence of their referents. But a residual ontological implication still remains: those who *read* mathematical statements literally are still committed to the existence of mathematical entities. To avoid such ontological commitment, Hellman offers a way of paraphrasing mathematical discourse that precludes reference to any abstract entity.

Hellman's paraphrasing strategy starts with a structuralist conception, similar to Shapiro's and Resnik's. However, where the latter see structuralism as giving reasons to postulate the existence of structures and their positions, Hellman sees structuralism as allowing the nominalist to get rid of commitment to non-spatiotemporal entities altogether. His strategy is smart. Since any object can perform the role of a number, we do not need to select any particular object or position in a structure to call, say, *the* number one. Rather, we can simply say that, *if* a natural-number structure existed, there *would be* a number one (Hellman, 1989, Ch. 1).

In this way, he proposes the reformulation of statements about *actual* structures into statements about *possible* structures. For arithmetic, he provides a translation pattern that converts any sentence *S* in the language of arithmetic into a modalized sentence. Instead of asserting *S simpliciter*, the nominalist asserts: "if there were any natural-number structure, *S* would hold in it." The formalization of this translation pattern is made in two steps. First, we have the schema:

$$\Box\forall X(X \text{ is a natural-number structure} \rightarrow S \text{ holds in } X)$$

This is not sufficient, though, since the possibility of there existing a natural-number structure must be secured. Otherwise the antecedent of the conditional in the formula above would be necessarily false, and hence any *S* would yield a true modalized sentence. Therefore, besides embedding arithmetical statements into the schema above, the nominalist has to assert:

$$\Diamond\exists X(X \text{ is a natural-number structure})$$

Summing up: for every arithmetical statement *S*, the nominalist refrains from asserting *S* simpliciter and, instead, asserts:

$$\Box\forall X(X \text{ is a natural-number structure} \rightarrow S \text{ holds in } X)\wedge \\ \Diamond\exists X(X \text{ is a natural-number structure})$$

By applying this translation schema, the nominalist manages to preserve the truth of arithmetic without committing to the actual existence of any number or structure. In this regard, Hellman's modal-structuralist reconstruction of mathematics[3] provides a real gain: it shows that the postulation of the existence of mathematical objects and structures is, after all, dispensable, even if the truth of mathematics is held to be indispensable.

However, it is easy to see that Hellman's account is intentionally inconclusive about the existence of mathematical entities; he admits that they *may* exist, as the formula above illustrates, but he does not provide any clue about their *actual* existence or nonexistence. As a nominalist, Hellman is better seen as preaching to the converted. His only aim is to provide for the nominalist a way to avoid ontological commitment to mathematical entities, but he does not offer any argument to the effect that nominalism is more faithful to the facts than platonism. In contrast to the accounts discussed in the previous sections, Hellman does not even provide an untestable hypothesis about whether there is any reality underlying mathematics.

1.5 Bueno's fictionalisms

Nominalist accounts do not make claims about a non-spatiotemporal realm, and therefore do not face the challenge of having to provide *direct* evidence that their claims are true about

[3] In Hellman (1989), he also provides a modal-structuralist reconstruction of set theory, allowing for the application of his account to mathematics in general.

an *inaccessible* reality. However, this is not an automatic advantage for the nominalist insofar as the nonexistence of mathematical entities is still considered to be an objective fact. Hellman's account proved to be incapable of shedding light on this issue. Can fictionalism be more successful in this task? Judging from Bueno's characterization of fictionalism, the prospect is discouraging.

> What is the difference between *fictionalism* and *nominalism*? As developed here, fictionalism is an *agnostic* view; it doesn't state that mathematical objects don't exist. Rather, the issue of their existence is left open. Perhaps these objects exist, perhaps they don't. But, according to the fictionalist, we need not settle the issue to make sense of mathematics and mathematical practice (Bueno, 2009, p. 63).

Bueno understands nominalism, in turn, as the "skeptical view" that categorically denies the existence of mathematical objects. This does not correspond to Hellman's nominalism, given that he acknowledges that mathematical entities may exist, as we have seen above. Terminological issues aside, the significant point here is that Bueno's fictionalism is intentionally as irrelevant for those interested in the objective fact of the existence of mathematical entities as Hellman's nominalism.

Bueno's only point is to make a case for agnosticism. To this end, he introduces two fictionalist approaches to mathematics intended to show that it is possible to preserve the literal reading of mathematical statements while simultaneously remaining agnostic about the existence of mathematical entities.

The first approach builds on van Fraassen's (1980) constructive empiricism. Bueno suggests that mathematical entities can be treated in the same way that van Fraassen treats unobservable physical entities. According to van Fraassen, "X is observable if there are circumstances which are such that, if X is present to us under those circumstances, then we observe it" (van Fraassen, 1980, p. 16), and X is unobservable if it is not observable. Mathematical entities, conceived of as non-spatiotemporal objects, fit the definition of unobservables perfectly. Since, according to van Fraassen, scientific theories are only required to be true about the observable features of the world they describe, the empiricist may remain agnostic about the truth value of scientific statements about unobservables. According to Bueno, the same applies to mathematical statements embedded in scientific theories.

> The crucial idea of the empiricist fictionalist strategy to applied mathematics is to insist that (applied) mathematical theories need not be true to be good. They only need to be part of an *empirically adequate package*. … The *whole package* is never asserted to be true; it's only required to be empirically adequate, that is, to accommodate the observable phenomena (Bueno, 2009, p. 65).

Now, because the mathematics recruited by scientific theories does not need to be true, the empiricist does not need to commit to the existence of the entities presupposed by the literal reading of mathematical statements. In this approach, singular terms occurring in mathematical statements still intend to refer to some object, but we simply do not know, and also should not care, whether the reference is felicitous or not. "In the end, the *existence* of unobservable objects is *not* required to make sense of scientific or mathematical practice. As a result, unobservable objects can be taken as *fictional*" (Bueno, 2009, p. 65).

Bueno's other fictionalist approach builds on Amie Thomasson's treatment of fictional characters (Thomasson, 1999) and Azzouni's treatment of logical quantifiers (Azzouni, 1997). Bueno starts with the idea that mathematical entities are similar to fictional characters in that both are kinds of *abstract artifacts*. According to Thomasson (1999, p. 35-36), an abstract artifact is "an object created by the purposeful activity of humans" that lacks spatiotemporal location. Being human creations, abstract artifacts are dependent on a creator and, according to Thomasson, they remain in existence only as long as there are copies of the works that describe them (or memories in someone's mind, or some other register) and a community of interpreters who are able to understand these registers.

> Similar points apply to mathematical entities. First, these entities are also *created*, in a particular context, in a particular time. They are *artifacts*. Mathematical entities are created when comprehension principles are put forward to describe their behavior, and when consequences are drawn from such principles. Second, mathematical entities thus introduced are also dependent on (i) the existence of particular copies of the works in which such comprehension principles have been presented (or memories of these works), and (ii) the existence of a community who is able to understand these works. It's a perfectly fine way to describe the mathematics of a particular community as being lost if all the copies of their mathematical works have been lost and there's no memory of them (Bueno, 2009, p. 71).

Bueno does not explain why it is "perfectly fine" to describe mathematical entities as analogous to fictional characters in that they are created and can be lost. This claim seems to go beyond agnosticism; it is a genuine hypothesis about the mode of existence of mathematical entities.

To regain agnosticism, Bueno recruits Azzouni's (1997) distinction between *quantifier* and *ontological* commitment. Azzouni claims that we can assert an existentially quantified statement without committing to the existence of any object, because ontological commitment only takes place when we explicitly assert that the entities we quantify over fulfill necessary and sufficient conditions for existence. For example, if we take observability to be a necessary and sufficient condition for existence, we may assert both that (a) 'There is a fictional detective called Sherlock Holmes' and that (b) 'Sherlock Holmes does not exist,' without contradiction. In (a), we incur quantifier commitment to a candidate entity. In (b), we deny that this candidate entity meets necessary and sufficient conditions for existence.

In this account of quantification, the only thing the fictionalist has to do to remain agnostic about the existence of mathematical entities is to lay down *only* sufficient conditions for existence. One such condition may be access: objects to which we have visual access, for example, can be said to exist. Abstract artifacts such as mathematical entities clearly do not meet this condition.

> But recall that these are only sufficient, and not necessary, conditions. Thus, the resulting view turns out to be agnostic about the existence of the mathematical entities the platonist takes to exist … The fact that mathematical objects fail to satisfy some of these conditions doesn't entail that these objects don't exist. Perhaps these entities do exist after all; perhaps they don't. What matters for the fictionalist is that it's possible to make sense of significant features of mathematics without settling this issue.

Agnosticism is not a satisfactory position for those who want this issue settled, but it would be a comfortable position in the meantime. The problem is that agnosticism comes with a price. In Bueno's two fictionalist approaches, mathematical statements turn out to have unknown truth values. For instance, take the sentence (c) 'There are infinitely many prime numbers.' As Bueno (2009, p. 75) puts it, (c) "is true as long as there are infinitely many prime numbers." But since the fictionalist does not know whether numbers exist, he does not know the truth value of (c) either. Recall that in Hellman's reconstruction, (c) is translated into the following true statement: 'if there were a natural-number structure, there would be infinitely many primes in it.' Before adhering to fictionalism, then, the anti-platonist needs to weigh what is more important: to preserve the truth of mathematical statements, or to preserve their literal reading.

Nevertheless, it is not clear that Bueno's approaches do preserve the literal reading of mathematical statements. The problem is brought to the fore when we attempt to figure out what the epistemic content of mathematical statements is in Bueno's versions of fictionalism. Knowledge requires truth, but in his approaches mathematical statements have unknown truth values. How can knowledge be possible in such an agnostic scenario?

According to Bueno, in the van Fraassen-inspired proposal, the fictionalist can claim knowledge of how the world *may be* if mathematical existential claims are true (Bueno, 2009, p. 67). For example, the fictionalist cannot claim knowledge of (c) taken literally, because numbers are unobservable. But, Bueno argues, he can claim knowledge of (c'): 'If there were numbers, there would be infinitely many prime numbers.' This closely resembles Hellman's modal reinterpretation of the same statement. Since some rephrasing is required in order to grasp its epistemic content, it seems that Bueno's fictionalism also involves some alteration of the reading of mathematical statements.

His other fictionalist strategy also involves rephrasing. In this approach, "knowledge of mathematical entities, just as knowledge of fictional entities in general, is the result of producing suitable descriptions of the objects in question and drawing consequences from the assumptions that are made" (Bueno, 2009, p. 73). Thus, mathematical knowledge amounts to knowledge of what deductively follows from what. This, however, should require the prefixation of every mathematical statement with a fictional operator. Thus, (c) should become (c"): '*In arithmetic*, there are infinitely many prime numbers.' Bueno is aware of this, but he thinks that it is no alteration, since a *covert* fictional operator was always there:

> The fictionalist is not introducing a fiction operator to mathematical statements. The statements are used in the context of principles that characterize the properties of the relevant mathematical objects. In this sense, the fiction operator—in the form of the comprehension principles that specify a certain domain of objects—is already in place as part of mathematical practice. The fictionalist is not adding a new item to the language of mathematics. Properly conceptualized, the fiction operator is already there (Bueno, 2009, p. 76).

Perhaps Hellman could also claim that, properly conceptualized, a covert modal operator was always there. In this case, we have distinct factual claims about mathematical language. Which is right? How can we figure out whether mathematical sentences really have such hidden operators? Bueno suggests that the use of such operators is "part of mathematical practice." If this is so, the confirmation of this hypothesis seems to require an empirical investigation of mathematical practice, which Bueno does not do. For now, let us take these claims as yet unconfirmed hypotheses about mathematical language, which cannot be investigated on the basis of purely a priori methods.

1.6 Leng's fictionalism

In contrast to Bueno's, Leng's fictionalism does not lead to agnosticism. Leng categorically denies the existence of mathematical entities. Her argument for the nonexistence of mathematical entities involves three steps: first, the rejection of the indispensability thesis; second, the interpretation of mathematics as a useful fiction; third, the invocation of Ockham's razor. Let us see each in turn.

Leng's point against the indispensability thesis as a reason to assume platonism is that the confirmation of scientific theories is not holistic, as Quine holds. According to her, the falsehood of confirmational holism is a fact of scientific practice:

> taking a closer look at the kind of theoretical statements that are generally considered as receiving confirmation from our theoretical successes, it appears that scientists do not in general take all the hypotheses of their best empirical theories to be equally confirmed by their theoretical successes (Leng, 2010, p. 9).

An example of this is the use of idealizations in science. Idealizations are literally false or empirically unsupported assumptions that, despite their falsehood, play an important role in simplifying or even making possible some scientific calculations. In fluid mechanics, for example, fluids are idealized as continuous substances because the mathematics involved in dealing with this idealized scenario is more tractable than the mathematics that would be required to deal with the realistic scenario wherein fluids are made up of discrete molecules. Now, "from the successful application of this hypothesis all that is concluded is that the hypothesis of continuity is good enough, not that it is true" (Leng, 2010, p. 112). The same goes for the mathematics applied in scientific theories, or so she argues. The confirmation of scientific theories as a whole need not be seen as implying the *truth* of the mathematics involved; mathematics may be no more than a *useful fiction*.

Leng's account of mathematics as fiction builds on Walton's pretense theory of fiction (Walton, 1990). For Walton, fictional stories are similar to games of make-believe in that both involve prescriptions to imagine. In one of Walton's examples, children engage in a game of make-believe where they pretend that stumps in a forest are bears. Naturally, bears in this child's game are only imagined, they do not exist in any way. Analogously, in fictional stories such as Conan Doyle's novels, readers are prescribed to *imagine* that there is a detective called Sherlock Holmes who lives on Baker Street. To Walton, Sherlock Holmes, just as the bears in the child's game, does not exist. As a result, in Walton's account, fictional propositions are neither false assertions about the real world nor true assertions about a fictional, abstract world. Rather, they are an invitation to engage in a game of make-believe, where the author and the reader *pretend* that those propositions are true.

For Leng, a similar process can explain the role of mathematics in science. According to her, mathematical posits in scientific theories are analogous to the bears in the child's game.

For example, the idealization that fluids are continuous can be seen as "a prescription to imagine of *real fluids* that they are continuous substances" (Leng, 2010, p. 159), just as in the game, where children are prescribed to imagine of stumps that they are bears.

This kind of pretense is useful because statements asserted in the context of a make-believe game can convey true information about the world. For example, when a child in the make-believe game says "there is a bear at the top of the hill," this statement, though literally false, conveys the information that there is a stump at the top of the hill. Leng claims that the same happens with idealized uses of mathematics in science, just like literally false statements about continuous fluids in fluid dynamics convey true information about real discrete fluids.

Even in cases where no idealization is involved, the role of mathematics in modeling physical phenomena may be accounted for in the same way. "Thus, to take an elementary example, taking my fingers as objects, it is *fictional* … that there is a 1-1, onto function f from the *set* of fingers of my left hand to the *set* of fingers on my right" (Leng, 2010, p. 178). The literal truth of this statement would require the existence of functions and sets. However, by conceiving of this statement as fictional, rather than true, we have the same epistemic benefit: the prescription to imagine that there is such a 1-1 function conveys true information about her fingers, viz., that they can be arranged pairwise.

Once mathematical propositions are seen as useful prescriptions to imagine, it is time to invoke Ockham's razor:

> if we can account for our successful scientific practices *without* assuming that our mathematically stated empirical theories assert truths about mathematical objects, then this provides us with a positive reason to *reject* the claim that there are any mathematical objects. For, although we cannot conclusively *prove* that there are no mathematical objects, and although our uses of mathematics are *consistent* with the possibility that there are mathematical objects satisfying the existentially quantified claims of our mathematically stated empirical theories, adopting our ordinary scientific standards of inquiry surely requires us to adopt the principle of Ockham's razor, according to which we ought not to multiply entities beyond necessity … Thus, I conclude, adopting a naturalistic trust of our ordinary scientific methods of confirmation requires us to reject the existence of mathematical objects (Leng, 2010, p. 259-260).

By handling Ockham's razor, Leng asserts an objective fact about reality: there is no thing we are prepared to call a mathematical object. This is a negation of platonism and faces the same difficulty that haunts its competitor: reality can be uncooperative. Despite Leng's efforts to show that mathematics *can be seen* as a useful fiction, despite the methodological appropriateness of Ockham's razor, it may be that mathematical entities really exist out there and that mathematical statements are true about them. What evidence do we have to the contrary? Rejection of the indispensability thesis does not amount to evidence for the nonexistence of mathematical entities. In the passage from which I took the epigraph to this dissertation, Azzouni writes:

> Some nominalistically-inclined philosophers may invoke—at this juncture [i.e., after rejecting the indispensability thesis]—an a priori methodological principle, such as Occam's razor. The proponent of this distasteful maneuver has overlooked that we can't a priori dictate the ontology of the universe (Azzouni, 2015, p. 1148).

This is what Leng does. Although Azzouni is right in complaining that it is a "distasteful maneuver," the nominalistically-inclined philosopher is not to blame. If the platonist made *testable* existential claims, the anti-platonist could base her negative existential claims on a *direct* refutation of those claims. But the existential claims of the platonist cannot be refuted by any direct approach. As we have seen, they are carefully formulated so as to avoid any possibility of independent validation. Therefore, the only alternative for the nominalist is to advance an explanation of mathematics that dispenses with mathematical posits altogether and, then, recruit Ockham's razor to categorically deny that such entities exist. Against "distasteful" a priori maneuvers, only other "distasteful" a priori maneuvers will do. Consequently, we cannot expect a validation of Leng's account coming from independent evidence that mathematical entities do not exist.

However, we can still ask whether Leng's account is correct about *mathematical practice*. If it turns out to be correct, this means that, in fact, mathematicians are concerned with creating useful fictions, rather than describing an independent reality of non-spatiotemporal entities, and this can count as evidence (though still indirect) for the nonexistence of mathematical entities. But Leng denies that her account is intended to be correct about mathematical practice. Indeed, she holds that *hermeneutic* fictionalism, i.e., the position according to which fictionalism is a correct interpretation of what mathematicians do, "is not supported by the evidence of mathematical practice" (Leng, 2005, p. 277). If this is so, there is no hope of confirming her negative existential claims via evidence from mathematical practice.

But then what is the point of Leng's account? Burgess suggests that fictionalisms that are not hermeneutic must be *revolutionary*. Revolutionary fictionalists "concede that their reconstructions of mathematics are not analyses of current mathematics, but amendments to it; not exegeses, but emendations" (Burgess, 2004, p. 23). In this case, Leng would be suggesting that, although mathematics as currently done does not involve make-believe games, it *should* be done in this way. Burgess thinks that revolutionary fictionalism is untenable: "given the comparative historical records of success and failure of philosophy on the one hand, and of mathematics on the other, to propose philosophical 'corrections' to mathematics is *comically immodest*" (Burgess, 2004, p. 30). Leng sees her fictionalism as revolutionary, but denies that she is immodestly proposing any correction to mathematical practice.

> The revolutionary fictionalist is not, after all, advocating the *abandonment* of mathematics, even though, according to fictionalism, the correct understanding of mathematical assertions is not as literal assertions of truth. … There is room, then, for a weak form of modesty, according to which we hold back from advocating revisions in successful practices that we do not believe to be truth-stating, by seeking to explain how such practices may be successful even if practitioners are involved in making assertions that we do not believe are literally true (Leng, 2005, p. 282-283).

Thus, her point is not that mathematics should change, that mathematicians should stop making existential claims and instead start making prescriptions to imagine. For Leng, nothing has to change in mathematical practice; only our way of understanding what is going on in mathematics "from the outside" should change (Leng, 2010, p. 26). "The result is a bloodless revolution, since mathematical and scientific practice is left undisturbed, but a revolution nonetheless" (Leng, 2005, p. 292). In other words, what Leng is proposing is a philosophical explanation of mathematics, according to which "*regardless* of what mathematicians actually mean by their assertions, the best interpretation of the assertions of mathematical theories is as literally false, and at most true only in some fictional sense" (Leng, 2005, p. 280).

But what evidence do we have that Leng's fictionalism is "the best interpretation," or "the correct understanding" (as quoted above) of mathematical theories? Admittedly, her interpretation is possible, but there are many other possible interpretations, some of which we have seen in the above sections. In which aspects is Leng's fictionalism better or more correct than Bueno's fictionalisms or even platonism? What are the criteria for adjudicating between philosophical explanations of mathematics if compliance with mathematical practice is not one? Anyone who takes mathematical statements to be true will immediately reject fictionalism as a *correct* account of mathematics, let alone *the best* one. Even a nominalist like Hellman takes the preservation of the truth of mathematical statements to be an important desideratum (Hellman, 1989, p. 2).

In my view, there is a tension between Leng's intention to give a *correct* account of mathematics and her denial that her account is hermeneutic. Revolutionary fictionalisms are not classifiable as correct or incorrect, since they intend to change, rather than account for, a given state of affairs; they advance proposals. Leng is not recommending changes in mathematical practice, but she is still recommending changes in the way we interpret mathematical theories. Apparently, she wants us to change the way we interpret mathematical theories because her brand of fictionalism is *the correct* way of doing it. An account that avows to be correct must be correct *about something*. If mathematical entities do not exist, and if Leng's account is not intended to be correct about mathematical practice (which is what remains after the platonic realm of mathematical entities is dismissed), it is intended to be correct about what. exactly? Perhaps Leng could say that her fictionalist account is correct about how the world is, since it does not postulate the existence of nonexistent entities; but this would beg the question against platonism.

Leng's denial that her account is hermeneutic protects it against refutation, but also impedes its confirmation. Is this an *ad hoc* way of dismissing requests for independent evidence? If her account were hermeneutic, evidence to the effect that mathematical practice does involve prescriptions to imagine, even if to a limited extent, even if covertly, would count as a confirmation of Leng's fictionalism. But, given that her account is not hermeneutic, I cannot see any way it could be confirmed or refuted.

1.7 Thomasson's and Maddy's deflationisms

All the difficulties we saw above concerning the question of the existence of mathematical entities may be a sign that this question is unanswerable, or that it misses the point, or that it is meaningless. If this is so, we had better deflate the debate: leave this question behind and move forward in other directions. While they propose different strategies, this is roughly the point of Thomasson (2015) and Maddy (2011, ch. 3). Let us examine each in turn.

In Thomasson (2015), the question of the real existence of mathematical entities is rendered meaningless. To achieve this result, Thomasson endorses the Carnapian distinction between internal and external existence questions relative to a linguistic framework (Carnap, 1983). According to Carnap, only internal questions are meaningful and answerable.

For example, the question 'Is there a prime number greater than 10?' is straightforwardly answered within arithmetic with a "yes." On the other hand, the external question 'Does 11 really exist?' is thought of as meaningless. Thomasson explains why this is so. The premise is that a linguistic framework sets the rules of the use of its terms, which amounts to setting the *meaning* of these terms. Then, when terms are used within the framework, i.e., in accordance with their rules of use, they are meaningful. In this case, existence questions have straightforward answers because, to answer them, we need only apply the rules of use of the terms that occur in the question. Now, if one rejects a straightforward internal answer and still asks whether something *really* exists, then she is no longer using the terms in accordance with their rules, otherwise she would have accepted the straightforward internal answer. In this case, she is using terms externally, hence detached from their rules of use. "But if [she] attempt[s] to use the terms while severing them from these rules of use, [she] make[s] the terms meaningless, and the questions pseudo-questions" (Thomasson, 2015, p. 39-40).

Accordingly, in Thomasson's approach, we can infer the existence of numbers and other mathematical entities trivially from mathematical language, and there is no deep philosophical question about their *real* existence. The putative "deep philosophical question" is seen as a meaningless pseudo-question. If asked about the mode of existence of the entities implied by mathematical linguistic frameworks, Thomasson may answer with her *simple realism*: "we should simply say that such entities exist—full stop—and adopt a simple realist view of them" (Thomasson, 2015, p. 146). That is, we cannot say anything more about such entities than what is allowed by the rules of use of the linguistic framework. In particular, we cannot say that mathematical entities exist in a platonic realm, because this cannot be inferred from mathematical language.

Thomasson's simple realism is a direct consequence of the Carnapian distinction between *meaningful internal* and *meaningless external* existence questions. But what evidence do we have that this distinction is correct? Taken at face value, in ordinary language, external existence questions *are* meaningful. Indeed, it is easy to make sense of external existence questions even if one assumes Carnap's linguistic frameworks and the corresponding internal/external distinction.

Both Carnap and Thomasson acknowledge that external questions can be charitably interpreted as *pragmatic* questions about the benefits of adopting a linguistic framework. Pragmatic questions are still external but nonetheless meaningful. For example, in the question 'Is number language useful?', terms are being used outside of the linguistic framework of arithmetic; the language of arithmetic does not even contain the term 'useful.' If this question is to be meaningful, it must be asked within some other linguistic framework. Then, we must suppose that this question is asked within a broader linguistic framework—a metalanguage—that is able to refer to the language of arithmetic and, say, to theoretical virtues. Pragmatic questions are, then, internal questions of broader linguistic frameworks.

Now, once this is admitted, we can introduce a linguistic framework that is able to refer to number language, thing language, mental-state language, platonist language, and ask whether numbers are things, mental states, or platonic entities. All of these will be internal and, hence, meaningful questions; and they allow us to go far beyond Thomasson's simple realism. As long as adequate definitions and rules of use are provided, we can even rehabilitate the original external question 'Do numbers really exist?'. To this end, we need only introduce suitable rules of use for the term 'really.' For example, we can stipulate that, when 'really' occurs in a question of the form 'Do *X*s really exist?', the question can be read as asking whether *X*s are things, mental states, platonic entities, or none of these alternatives. Conclusion: even if Thomasson's Carnapian account of meaning is right, the ontological debate about the existence of mathematical entities is not deflated. Properly interpreted, originally external existence questions can be re-posed in higher-order linguistic frameworks that are able to refer to lower-order ones, in which case they become internal questions. And then all the difficulties involved in the philosophical debate about the ontology of mathematics are back.

The other attempt to deflate the debate about the existence of mathematical entities I consider here is Maddy's *thin realism*. Maddy claims that existence questions about mathematical entities should be, and in fact are, answered in mathematics itself. If the philosophical concern about the ontology of mathematics is motivated by a suspicion that those things whose existence is proved in mathematics might actually not exist, Maddy claims that this suspicion is unfounded. According to Maddy's naturalism or *second philosophy*, "a successful enterprise, be it science or mathematics, should be understood and evaluated on its own terms, [and] should not be subject to criticism from … some external, supposedly higher point of view" (Maddy, 1997, p. 184). For Maddy, when philosophers ask for the justification of mathematical methods, or whether mathematical entities really exist, they are violating this naturalist recommendation. *Thin realism*, by contrast, is a philosophical position that takes the second philosopher's mandate to judge mathematics in its own terms seriously.

Maddy formulates thin realism as a philosophical position about the ontology of set theory. The thin realist lemma is: "sets just are the sort of thing set theory describes; this is all there is to them; for questions about sets, set theory is the only relevant authority" (Maddy, 2011, p. 61). Hence, since the existence of sets is explicitly asserted and proved in set theory, sets exist—full stop. Whether sets exist independently of us as eternal platonic entities or are our creations is not a proper question in set theory, and, therefore, this is not a proper question at all. This is what makes Maddy's realism *thin*: "[i]n contrast with the entities posited by the various rich metaphysical and epistemological theories of the Robust Realists—which all go well beyond 'the positive things asserted by set theory'—these sets will seem rather insubstantial" (Maddy, 2011, p. 62). In Maddy's view, those who ask for further evidence (other than the evidence provided by mathematics itself) that mathematical entities exist are posing an undue challenge to a successful rational practice:

> if the Robust Realist is right, if the goal of set theory is to describe an independently-existing reality of some kind, then it appears that Cantor's evidence needs supplementation, and not supplementation of the same sort, like adding in Dedekind's grounds and so on, but supplementation of an entirely different kind: we need an account of how the fact that sets serve this or that particular mathematical goal makes it more likely that they exist. Without this account we have no way of ruling out the possibility that reality is sadly uncooperative, that much as we'd like to use sets in our mathematical pursuits, they just don't happen to exist. To the Second Philosopher, this hesitation seems misplaced: why should perfectly sound mathematical reasoning require supplementation? Hasn't something gone wrong when rational mathematical methods are called into question in this way? (Maddy, 2011, p. 58).

Thus, both Maddy and Thomasson agree that the question about the real existence of mathematical entities is misplaced. In contrast with Thomasson, though, Maddy does not see it as a pseudo-question. For Maddy, this question is completely meaningful. The problem is that it is asked from an epistemically inferior position—philosophy—towards an epistemically superior discipline—mathematics. At least, this is the so-called "naturalist" or "second-philosophical" assumption in action here.

If philosophers were attempting to philosophically answer a mathematical question about mathematics, I would agree with the Second Philosopher. We had better leave this work to the mathematician. However, the philosophical question about the existence of mathematical entities *is not* a mathematical question; it cannot be answered with a mathematical proof. As I pointed out above, this question can be understood as asking whether mathematical entities can be identified as things in the physical world, or thoughts in our minds, or things in a platonic realm, or something else, or nothing at all. We cannot expect an answer from mathematics to this question, because this is not a question about mathematics alone; it is a question about the relationships between mathematical posits and other kinds of things or posits.

Questions about the relationships between apparently different kinds of mathematical entities are commonly asked within mathematics itself. A classic example: can numbers be sets? This question cannot be answered in arithmetic alone. If we adopt a position like thin realism and assume that "numbers just are the sort of thing arithmetic describes; this is all there is to them," we have no means to address this question. But by taking arithmetic and set theory together, and doing some philosophy, we can formulate an answer (Benacerraf, 1983b). It would be quite arbitrary to forbid this question by decreeing that "numbers just are the sort of thing arithmetic describes," without providing compelling reasons to deny that they are sets. In the same way, asking about the relations between sets and, say, physical things does not challenge the adequacy of set theory to address set-theoretic questions about sets. Conclusion: surely philosophy is epistemically inferior to mathematics when it comes to mathematical matters, but this has no impact on what *philosophical* questions philosophers can pose about mathematics.

Thomasson's claim that external questions are meaningless and Maddy's claim that mathematics cannot be critically questioned from a philosophical point of view can be seen as belonging to the same supply of *ad hoc* maneuvers intended to ban requests for independent evidence for existential claims. These maneuvers can protect the accounts that adopt them from refutation, but also make them unverifiable. How could Thomasson and Maddy *demonstrate* that mathematical entities simply are what mathematics says about them without resorting to extra-mathematical reasons?

1.8 Conclusion

A priori speculation about the existence of numbers and other mathematical entities may be a good way of putting forward hypotheses, but it is certainly not the proper way of verifying them. In this regard, the most one can do by speculation only is to advance philosophical theses or methodological principles intended to show that no independent verification would be needed. However, as I argue in this chapter, such theses and principles can be seen as *ad* *hoc* maneuvers intended to ban requests for independent evidence. Except for Hellman's and Bueno's approaches, which overtly refuse to assert anything objective about the existence or nonexistence of mathematical entities, all the other approaches considered here make use of these maneuvers to some degree.

Although the approaches discussed here do not exhaust all a priori strategies that may be available to philosophers, it is quite unlikely that some other a priori procedure could perform better than those examined here. Deductive formal proofs are the most effective and reliable a priori methods we have, the existence of mathematical entities is provable in this way, but philosophers do not take these proofs at face value. Even deflationists feel pressed to supply these proofs with further arguments for their conclusions. Why would less reliable a priori methods do better?

If my arguments are correct, the prospects for investigating the metaphysics of mathematics by purely a priori methods are discouraging. On the other hand, if it is the case that, if mathematical entities exist, they exist outside of space and time, the prospects for investigating the metaphysics of mathematics by means of a posteriori methods are even more discouraging. But we do not need to assume that this is the case. Indeed, we *should not* assume this. The very claim that mathematical entities, if they exist, must necessarily be non-spatiotemporal is dependent on philosophical accounts for which we do not have any conclusive evidence. Thus, we can leave open the question of the nature of mathematical entities, in case they exist, and try to make progress with a posteriori methods.

The inconclusiveness of a priori methods for investigating the metaphysics of mathematics can be seen as progress: it shows that, if we want more conclusive results, we need to broaden the scope of the investigation. A starting point for this broader investigation may be an inquiry into how we, human beings, actually acquire and produce mathematical knowledge. I start this investigation in the next chapter.

Chapter 2

A methodological alternative: an empirically informed approach to numbers

A much discussed topic in the contemporary literature on the philosophy of mathematics is the so-called Benacerraf's problem (Benacerraf, 1983a). The problem consists of explaining how human beings living in a spatiotemporal world can have knowledge of non-spatiotemporal mathematical entities. Although I am not particularly concerned with Benacerraf's problem here, referring to it helps illustrate the methodological approach I will adopt to investigate the ontological status of numbers.

Attempts to solve Benacerraf's problem usually involve denying that we need access to non-spatiotemporal entities in order to obtain knowledge of them (e.g., Shapiro (1997)) or denying that mathematical entities are non-spatiotemporal (e.g., Maddy (1990)). These approaches assume in advance both a premise about the mode of existence of mathematical entities and that there is arithmetical *knowledge*. From these assumptions, they go on to explain how knowledge of that kind of entity is possible.

What I am proposing here with respect to arithmetic is a third strategy, whose first step is to suspend judgment about: (a) the existence and the mode of existence of numbers, and (b) the existence of numerical knowledge. Because I am still assuming the literal reading of numerical statements, suspension of judgment about (b) is a consequence of suspension of judgment about (a). Knowledge requires truth. If we suspend judgment about the existence of numbers, we cannot assert whether numerical statements are true or false, and therefore cannot claim knowledge of them. Once we have suspended judgment about the existence of these things, what remains are *numerals* (sometimes occurring within sentences) and certain mental contents that are activated by them. These mental contents I call "number concepts," and what people say about these concepts and do by relying on them, I call "numerical competence" (instead of "numerical knowledge").

It is worth noting that here I am using the term 'concept' to refer to psychological entities. This is in line with experimental philosophers' approach to concepts. For example, Machery sees concepts as "bodies of information [people have] about individuals, classes, substances, or events" (Machery, 2017, p. 210). However, this should not be taken as implying that *numbers* are mental entities. Concepts as mental contents may still be about non-mental objects, including platonic entities (e.g., *my* concept of three may be about the platonic, non-spatiotemporal object three). In the course of this investigation we will find out whether number concepts are about platonic objects, physical objects, classes, substances, events, something else, or nothing at all.

With these caveats in place, we can start investigating how human beings acquire numerical competence *in real life*. Regardless of Benacerraf's challenge, which is aimed at mathematical *knowledge*, it is certain that people do have numerical *competence*, and therefore we can investigate how they obtain it. This investigation can follow a "reverse engineering" strategy: starting with the assumption that certain humans have numerical competence, go back to a time when they did not, and observe what happened in the meantime. This involves answering two questions: what does a human being experience during cognitive development so as to acquire numerical competence, and what, in the history of the human species, enabled us to acquire it? Hopefully, data about our experiences as individuals and as a species will suggest a hypothesis about what entities—objects, classes, substances, events, platonic entities, something else, or nothing at all—underlie our number concepts. In other words, what I am proposing is a bottom-up approach wherein we assume a neutral stance regarding the existence and nature of numbers and numerical knowledge and "let the data speak for itself," as it were. If this approach succeeds, at the end of this investigation we will have understood how numerical competence emerges, what numbers are (if anything), and whether and how numerical knowledge is possible.

The basis of this strategy is the belief that the fundamental phenomenon to be explained in an investigation of the metaphysics of mathematics is a class of human experiences. Humans do mathematics and use it to help them solve practical and theoretical problems. When investigating the metaphysics of mathematics, we are looking for a set of entities that can account for the main features displayed by these human experiences. Whether non-spatiotemporal entities are required to make sense of the arithmetical experience of human beings is something we will find out in the course of this investigation. This should never be assumed in advance.

Data relevant for this investigation comes from several sources, such as numerical cognition, linguistics, and mathematics education. Most of this data is reviewed in Chapters 3 to 6. In the first section of this chapter (section 2.1), I start reviewing data from numerical cognition that shows that numerical competence is inseparable from the use of symbolic systems for numbers; i.e., *numerals* seem to be indispensable for the acquisition of number concepts. The discussion in section 2.1 brings to the fore the instrumental role of numerals as *cognitive tools* and sets the direction for the rest of the investigation. Cognitive tools are an important topic in the psychological and philosophical literature on cognition. In sections 2.2 to 2.4, I review some topics from this literature that will inform the discussion in this and following chapters. In section 2.2, I discuss what cognitive tools are. In section 2.3, I present Vygotsky's (1978) concept of *internalization* and Menary's (2007b; 2015) concept of enculturation, which help explain how cognitive tools transform our brain and give us new abilities (such as numerical competence). In section 2.4, I explore the concepts of de-semantification and re-semantification (Dutilh Novaes, 2012; Krämer, 2003) to show how symbolic cognitive tools help us overcome cognitive shortcomings and obtain contents that we could not obtain without their aid (such as number concepts).

All this discussion of cognition and cognitive tools may seem odd, or even irrelevant, for those used to more traditional philosophy of mathematics, but I encourage the reader to stay with me until section 2.5, where I sketch a hypothesis about the existence and nature of numbers suggested by the data and theories reviewed in this chapter. This is the hypothesis I will further elaborate on and defend in the remaining chapters of this dissertation.

2.1 Quantical and numerical cognition

One of the most remarkable findings from numerical cognition is the fact that the most basic cognitive foundations of numerical competence are neither learned nor exclusively human. Human beings and many other animals share an inborn set of abilities to identify and discriminate between discrete quantities (Kadosh & Dowker, 2015). In human beings, for example, these abilities are already present in neonates during their first hours of life, long before any mathematical training could have taken place. Antell and Keating (1983) showed that, when presented with a collection of two items and then a collection of three items, 53-hour-old newborns can notice the numerical difference between them. Likewise, newly hatched chicks show surprising abilities. Rugani, Fontanari, Simoni, Regolin, and Vallortigara (2009) exploited chicks' willingness to be together with their siblings to investigate their ability to "count" and "calculate." They hid a chick's siblings behind one occluder and started moving some of the chicks, one by one, to behind another occluder. The tested chicks were allowed to see their siblings being placed behind the first occluder and then being moved to behind the other. When allowed to join their siblings, the tested chicks joined the larger group, which suggests that they were able to "count" how many of their siblings were moved and to "subtract" the resulting number from the number of chicks originally placed behind the first occluder. Fish, bees, and several other species have been shown to possess similar abilities (Agrillo, 2015). The fact that these abilities seem to be innate in all the studied species suggests that they have genetic evolutionary roots (Brannon & Merritt, 2011).

Another important finding is that these abilities are accurate only up to a point. For example, chicks tested in Rugani, Regolin, and Vallortigara (2008) were able to accurately identify the larger group only when groups consisted of one, two or three individuals. For larger groups (of four, five, or six individuals), their precision was lower. Human infants and other animals face a similar limitation. Above the limit of three or four items, the larger the quantities involved, the greater the numerical difference between them must be for the subjects to perceive a difference. For example, in Brannon and Terrace (2000), monkeys were more accurate when discriminating collections of four items from collections of nine items (a numerical difference of five), than collections of seven items from collections of nine items (a numerical difference of two).

In general terms, the abilities of infants and non-human animals to discriminate discrete quantities consist of the ability to *subitize*, i.e., to identify and distinguish the size of collections of up to three or four items quickly and accurately; the ability to *estimate*, i.e., to identify and distinguish the size of larger collections quickly but only approximately, demanding ever-increasing numerical differences so that the perceiver can notice them; and the ability to perform operations analogous to sums and subtractions over the sizes of col-

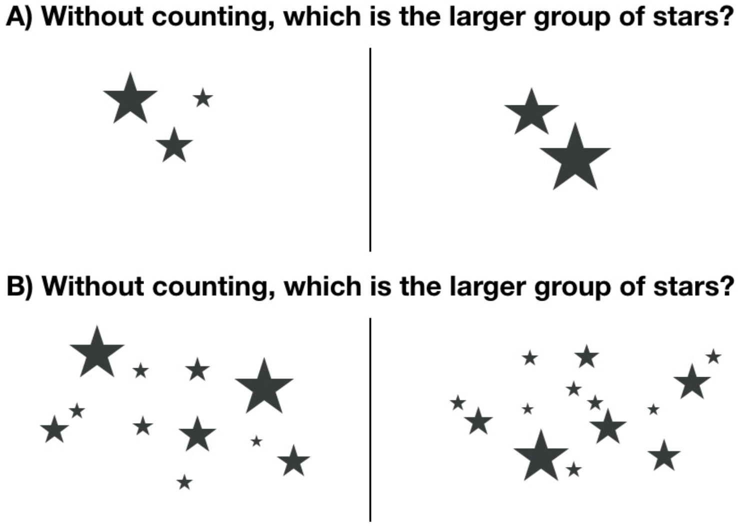

Figure 2.1: In (A), it is easy to tell which is the larger group at a glance with full certainty without counting. The ability to do so is called *subitizing*. In (B), without counting it is possible to guess, by *estimation*, which is the larger group, but certainty is obtained only by counting both collections and comparing the numbers.

lections perceived through subitizing and estimation. Naturally, human adults also display these abilities. Figure 2.1 provides two simple tasks intended to elicit the reader's abilities to subitize and estimate. I discuss these abilities in more detail in Chapter 3. For now, two points are important. First, these abilities are non-symbolic, since infants, monkeys, and chicks do not have at their disposal culturally created symbolic systems, such as number words, on which they could rely for counting. Second, these non-symbolic abilities face constraints that are not likely to apply to the analogous operations performed with the aid of culturally created symbolic systems.

Following Núñez (2017), I will call the set of inborn non-symbolic abilities shared by humans and non-human animals *quantical cognition*. In the literature on numerical cognition, these abilities are usually called *non-symbolic numerical cognition*, but for reasons that will be made clear in Chapter 3, the adjective 'numerical' is inadequate here. I will reserve the term *numerical cognition* to refer to the set of cognitive skills to deal with *symbolic* numerical information found in numerate humans (these abilities are usually referred to as "symbolic numerical cognition" in the literature). In addition to the role of symbols, numerical and quantical cognition are clearly distinguished from each other by performance indicators: numerical cognition is more likely to deliver exact and accurate outcomes no matter the quantities involved, but it demands longer response times that increase as the involved quantities become larger, whereas quantical cognition displays faster response times, but is accurate only up to the subitizing limit (around three), becoming increasingly fuzzy as quantities grow.

The relationship between quantical cognition and numerical cognition is not fully understood yet (I address their relationship in section 4.2), but it is believed that quantical cognition provides the genetically evolved preconditions for the development of numerical cognition in humans. However, as shown by the fact that non-human animals endowed with quantical skills do not develop numerical cognition, the mere presence of quantical skills is not sufficient for the emergence of numerical cognition. This is where the human ability to use culturally created symbolic systems comes in. The use of systems of numerals and other notations is believed to give us new capacities by extending and improving our innate quantical abilities.

A study conducted by Barth et al. (2006) helps further illustrate the difference between quantical and numerical cognition and the indispensable role of symbols in the latter. Barth and colleagues tested adults and five-year-old children on non-symbolic operations analogous to arithmetic operations performed over collections of physical objects. In symbolic arithmetic operations, the operands are conveyed in symbolic format—e.g., 2+5—and the agent is allowed to use all her competence with school arithmetic to calculate the result. In a non-symbolic operation, by contrast, the input is provided in the form of particular discrete quantities—e.g., arrays of dots presented on a screen—and the agent is not allowed to use symbolic resources to execute the operation, either because the agent does not know such resources (as in the case of infants and non-human animals) or because the experimental set up is especially designed for this purpose. For example, experimenters can present stimuli for a very short period of time, enough to allow estimation, but too short to allow counting. Figure 2.2 illustrates some of the experimental setups used in Barth et al. (2006).

As results presented in Barth et al. (2006) show, both adults and children are similarly able to perform non-symbolic operations, but adults' performance in operations with quantities above the subitizing limit fell short of what could have been expected if they had been allowed to count and calculate symbolically. For example, in the addition task (Figure 2.2A), adults succeeded in 75% of the trials; in the subtraction task (Figure 2.2C), adults succeeded in only 67% of the trials. For a rough comparison, LeFevre, DeStefano, Penner-Wilger, and Daley (2006) report that in symbolic subtraction, where the input was provided in Arabic digits and participants were asked to calculate mentally, adults succeeded in 92% of the trials. The difference in performance between non-symbolic and symbolic subtraction reflects the different degrees of accuracy of quantical and numerical cognition.

Barth and colleagues also found that some non-symbolic operations were more difficult than others, an effect consistent with the limitations of quantical cognition. Performance dropped significantly as the difference between the displayed outcome and the correct outcome decreased. For example, participants were less accurate in noticing that the result of adding 18 dots to 12 dots is not 26 dots (difference of four dots), than in noticing that the result of adding 20 dots to 15 dots is not 26 dots (difference of nine dots). Quantical estimates are fuzzy, and thus tend to overlap when the difference is relatively small. By contrast, by counting we are able to detect a difference of just one unit no matter the size of the involved collections, provided that time and expertise are available.

Another difference between counting and estimation is that, to estimate, we just look at the target collection as a whole and after a few milliseconds we guess its cardinal size, whereas to count we need to follow a systematic procedure, extended in time, governed by clear rules and involving an ordered sequence of words. It is surprising that in many cases, estimates that recruit only non-symbolic skills are approximately right, but it is clear that only by counting

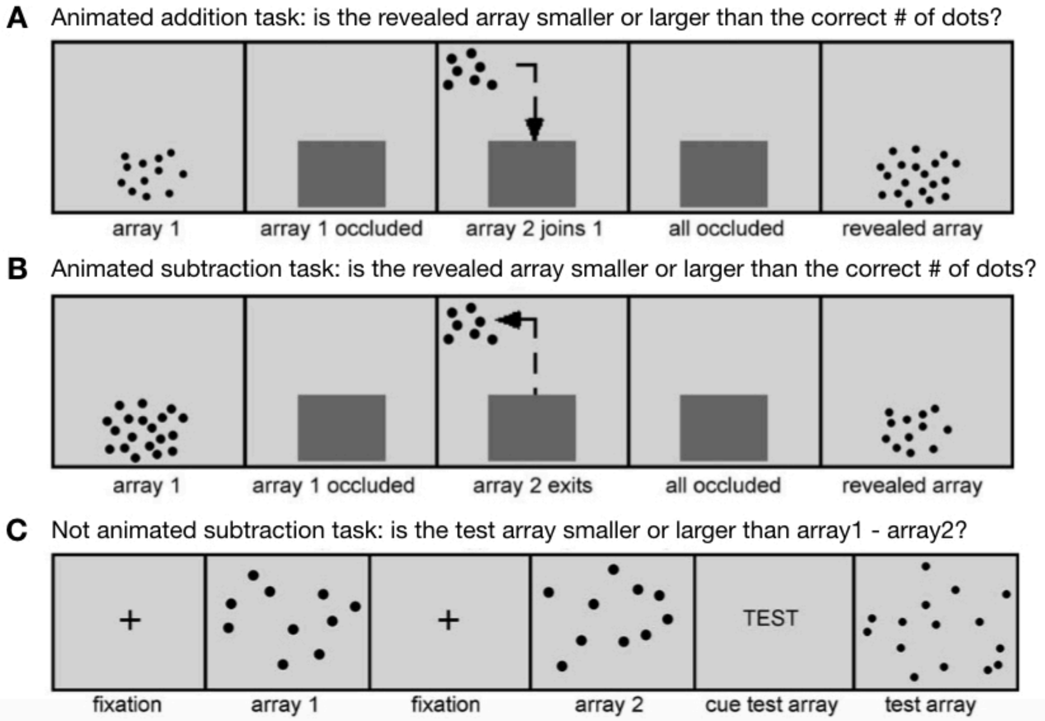

Figure 2.2: Stimuli used in Barth et al. (2006) to test adults' and children's ability to perform non-symbolic operations analogous to additions and subtractions. (A) In the animated addition task, participants were presented with an animation in which an array of dots was displayed for a short time and subsequently occluded by a panel. Then, another array of dots was moved to behind the occluder, joining the previously occluded initial array. When the occluder was removed, participants were asked whether the revealed array had fewer or more dots than expected. (B) The animated subtraction task was similar, but after the first array being occluded, a number of dots was removed from behind the panel. (C) In the not animated subtraction task, three arrays of dots were presented sequentially, and participants were asked whether the third array had fewer or more dots than the subtraction of the second array from the first. In the three tasks, stimuli in which the revealed array had exactly the correct number of dots were not tested. Children were tested only in the animated tasks, with smaller adaptations, such as the inclusion of engaging narration. (Figure adapted from Barth et al. (2006, p. 202, 205).)

can we accurately determine the cardinal size of any collection with more than three or four items. The same goes for operations such as additions and subtractions. Participants in Barth and colleagues' experiments just observed dots coming in and out from behind the occluder and guessed the outcome after a few milliseconds. If they were to calculate accurately, they would have to count the dots and add or subtract the results symbolically, a time-consuming complex operation.

These differences in overt behavior are taken to reflect differences in the way symbolic and non-symbolic operations are implemented in the brain. I address the cerebral implementation of quantical skills in sections 3.2 and 3.3. Next, I present a cognitive model of the implementation of numerical competence in the brain in which the instrumental role of symbols is made clear.

According to the Triple Code Model (Dehaene, 1992; Dehaene & Cohen, 1995, 1997), the leading model of mental arithmetic processing, the most reliable strategies for mental calculation depend on symbolic resources typically learned at school. The gist of the Triple Code Model is the claim that numerical information is encoded in the brain in three formats. Two of these formats directly mirror the two numeral systems we commonly use: a system of oral number words and their corresponding written forms (in English, 'one,' 'two,' 'three,' etc.) and the decimal place-value system of Arabic digits ('1,' '2,' '3,' etc). These two symbolic systems give rise to two modules of arithmetical processing in the brain: one module that contains representations of the visual forms of Arabic digits (let us call it M1); and one module that contains representations of both phonological and graphemic forms of number words and arithmetical facts encoded in verbal format, such as addition and multiplication tables (let us call it M2). Naturally, the presence of these modules and their contents are relative to cultural practice; people who did not learn, say, multiplication tables will not have them encoded in phonological format in M2; people who do not use Arabic digits will not have M1 (or will have a corresponding module encoding the visual form of the symbols they use). It is worth noting that, according to the model, in both M1 and M2 representations are *asemantic*, i.e., they do not have meanings stored together with phonological and graphemic forms. These modules store only verbal and graphic representations of the symbols themselves and phonological representations of the arithmetical facts conveyed in verbal format that we learn by heart.

Meanings are provided by the third module (let us call it M3). M3 encodes numerical information by means of analog representations of discrete quantities. These representations give us the cardinal values associated with Arabic digits and number words. In Dehaene and Cohen's original model, the exact cardinal values that we associate with number words and digits are believed to result from the sharpening of rougher, approximate innate representations of numerosities that are part of the mechanisms that implement quantical cognition. According to the model, what sharpens these vague numerosity representations, converting them into exact cardinal values, is the very number words and digits codified in M1 and M2 (more on this in section 4.4). In other words, it is the process of learning a numeral system that generates number concepts (exact cardinal values) in the brain by sharpening vague analog representations of discrete quantities innately provided by quantical cognition. There are other models of how we acquire number concepts, but in all of them (except for nativist accounts) they come from learning a numeral system (more on this in Chapter 4).

If the Triple Code Model is correct, the three sources of numerical information in the brain originate from culturally created symbolic systems, with a contribution from the innate mechanisms of quantical cognition for M3. That is, numerical cognition is a blend of quantical cognition plus culturally created symbolic systems. The cognitive function of symbols is to provide exactness and accuracy in operations involving collections of more than three or four items. For very small collections, the quantical ability to subitize suffices; for larger collections, though, if we want the same levels of exactness and accuracy, we need to make use of techniques involving numerals and other symbolic resources. This is illustrated in task B of Figure 2.1: the exact number of stars in each group can be found only by counting. In this sense, the counting procedure functions as a cognitive tool that enables us to regain the accuracy of subitizing in the evaluation of the size of collections that otherwise we could only estimate. According to Dehaene and Cohen's model, over time, practice with counting "sharpens" the approximate representations of numerosity we recruit for estimation and give us the exact cardinal values we associate with numerals. (Notice that this departs from the traditional view according to which the primary function of symbols is the communication of previously given contents. Here, symbols themselves give rise to the contents that they communicate. I elaborate on this point in section 2.4.)

More generally, the suggestion is that the symbolic systems of school arithmetic are cognitive tools that allow us to overcome the limitations of innate quantical skills, thus obtaining truly numerical competence. It is surprising that animals from fish to monkeys share with human infants abilities such as subitizing; but it is also surprising that, even sharing these abilities, only humans are able to display truly numerical competence. Between an infant whose capacity to discriminate discrete quantities accurately, although remarkable, is limited to three or four items, and a child at school age who is able to count up to more than one thousand, there is a long period of training in culturally created techniques. The child was born with some capacity to deal with quantities; training in these techniques launched her from those limited capacities to truly numerical competence.

In the "reverse engineering strategy" I propose in the introduction to this chapter, this suggests that these culturally created techniques, as well as inborn quantical skills, may be key elements in the ontology underlying numerical competence. This suggestion is one of the central claims behind the ontological hypothesis I formulate in section 2.5. The overview of the role of symbols in numerical cognition given above was intended to provide initial evidential support for this claim. More evidence for this will be presented in Chapters 3 and 4. In the next section, I introduce the concept of a cognitive tool.

2.2 Cognitive tools

Hand tools, such as hammers, shovels, and saws, are prototypical examples of tools. They are human creations whose primary function is to facilitate or enable the performance of certain operations. They do so by enhancing, extending, or adding new features to the innate, genetically evolved power of our hands. For example, the genetic configuration of our hands and arms is such that we are able to dig a hole in beach sand without using tools. However, bare hands are too fragile to accomplish digging harder soils. By using a shovel, though, we are able to overcome this limitation. The shovel's hard blade enables us to cut into the soil, and the shovel's handle amplifies the force of our arms, making it easier to lift and toss the shovel's content. The shovel adds to our genetically developed skills features that we lack: hardness, sharpness and incremented force.

Cognitive tools are like hand tools in many aspects. They are also culturally created instruments that facilitate, improve or enable the performance of certain operations for which our "bare brains" are limited or are not completely suitable. Think of an abacus, a prototypical example of a cognitive tool. An abacus enables its user to perform arithmetical operations faster and more accurately than she could by relying only on mental operations. Like a shovel that enables its user to dig in harder soils, an abacus enables its user to calculate with larger, and therefore harder,[1] numbers. Cognitive tools, just like hand tools, build upon genetically evolved features of our body to enhance or extend our cognitive skills. Abacuses, like hammers and screwdrivers, are designed to exploit the human capacity to make precise movements with hands and fingers. Cognitively, abacuses rely on our innate quantical skills. Users of abacuses can move the correct number of beads fast and accurately because they subitize. They do not count "one, two, three" to move groups of one, two or three beads; they just "see" the number of beads they are moving.

Despite these similarities, however, cognitive tools are unlike hand tools in a key aspect: they do not need to be three-dimensional, handleable external objects. Cognitive tools can be *techniques*, i.e., rule-based methods of performance that facilitate the fulfilment of cognitive tasks. Usually these techniques involve the use of symbols, which can be manipulated either on external media (e.g., on paper) or mentally. What justifies calling these techniques "tools" is that, similar to handleable external tools, they are human creations that boost our capacities.[2]

Take, for example, shopping lists, an oft-cited example of a cognitive tool. The chief function of shopping lists is to aid memory. Various aspects of shopping lists contribute to the fulfilment of this function, especially the enduring character of their materials (e.g., ink and paper). But another central aspect of shopping lists is that they carry symbols inscribed on paper. A blank piece of paper, or one whereupon only random marks are inscribed, is not a shopping list. In order to fulfil its role as a memory aid, a shopping list must contain inscriptions produced in accordance with a system of codification previously known by its users, so that they can correctly and easily interpret the marks. In this sense, the writing system—a technique for encoding information in permanent media—used in a shopping list counts as a cognitive tool in its own right. Writing is a technology *per se*, which can be combined with other elements to produce other cognitive technologies, such as shopping lists, books and smartphones.

By the same token, symbolic systems for numbers, including verbal lists of number words, can be seen as cognitive tools in their own right. These systems can be used in com-

[1]As a rule, calculations involving larger numbers are harder. In the literature on numerical cognition, this is called the *problem size effect*: "people take longer and make more errors to solve problems like 9 + 7 = 16 with large digits and large answers than to solve problems like 2 + 3 = 5 with small digits and small answers" (Zbrodoff & Logan, 2005, p. 331).

[2]For definitions of cognitive tools in accordance with the way I am using the term here, see Dascal (2002), who defines cognitive technologies as "every systematic means—material or mental—created by humans that is significantly and routinely used for the performance of cognitive aims," and Heersmink (2013).

bination with other elements to produce more obvious cognitive tools, such as abacuses and calculators, but the very list of number words, or the notation system of Arabic digits, qualifies as a cognitive technology in itself. We make use of number words as cognitive tools, for example, when we recite them in ascending order to determine the cardinal size of a collection—in other words, when we count. The same goes for the exploitation of the properties of the decimal place-value system of Arabic digits for mental or pencil-and-paper calculations. These symbolic systems provide rule-based methods of performance that not only facilitate, but indeed make possible exact calculations and the determination of the exact cardinal size of collections with more than three or four items. As hand tools do with respect to our hands, these techniques enhance, extend, and add new features to the innate, genetically evolved power of our brains.

Another difference between cognitive tools and hand tools is that some cognitive tools can be internalized, whereas hand tools cannot. By repeatedly using a saw, a carpenter progressively becomes more skillful at sawing, making the process faster, more efficient, and more precise, but the carpenter will never acquire abilities that will allow her to dispense with external saws. By contrast, when it comes to symbolic cognitive tools, repeated use can have the effect of enabling the user to simulate the operation of the tool in her mind. Thus, under certain conditions the user may replace the use of external instantiations of the tool by mental simulations in order to perform operations which, initially, she could perform only with the help of the tool in its external format. One example is the decimal place-value system of Arabic digits. It is a systematic means created by humans to represent numbers and facilitate cognitive tasks such as calculating with pencil and paper. Initially we learn the decimal place-value system as an external means of representing numbers and calculating on paper. Over time, however, we internalize the system and start to use it in mental calculations. Notice that this does not amount to becoming able to dispense with the use of the tool; only its external use becomes dispensable (for the execution of easier operations, at least). As mentioned above, mental calculations still rely on brain mechanisms that encode Arabic digits. Internalization of cognitive tools is possible because our brain, as opposed to our hands, is *plastic*: it can be reshaped through learning (more on this in section 2.3).

Numeral systems illustrate a characteristic that cognitive tools share with other kinds of tools. There are certain tools, such as saws, that are indispensable for certain activities. We cannot saw a board without the aid of a saw. Under certain conditions, we can tear or break a piece of wood by using only our hands, but to saw it, we need a saw. The same happens in the cognitive realm: to count and to calculate precisely, we need numerals; to predict solstices and equinoxes, we need calendars; to find very high prime numbers, we need computers. With respect to this, Dascal (2002) distinguishes "constitutive" from "non-constitutive" cognitive tools. Constitutive cognitive tools are those "such that without them certain cognitive operations cannot be performed," whereas non-constitutive cognitive tools, "although extremely useful for the facilitation of the achievement of certain cognitive aims, are not a sine qua non for that" (Dascal, 2002, p. 41). While numeral systems, calendars, and computers are cognitive tools constitutive of certain cognitive tasks, a shopping list with only ten items is not constitutive of the activity of remembering what we need to buy. The list facilitates remembering, but we can do without it.

In the literature on the extended mind thesis, the adjective 'constitutive' has been used to mean that external devices are an integral part of the cognitive processes in which they are used, which has led to intense debate (see Kirchhoff (2014)). I do not want to take sides on this issue. The important point here is that there are certain tools (or certain classes of tools) whose use is indispensable if we are to fulfill a certain cognitive task.

To sum up, cognitive tools are human creations that aim at facilitating or enabling the execution of certain cognitive tasks. They do not need to take the form of three-dimensional devices; they may be rule-based methods of performance. They may be indispensable for the completion of a cognitive task. And, most remarkably, as opposed to hand tools, some cognitive tools can be internalized.

2.3 Internalization

Internalization, as I am using the term here, refers to the process through which an agent acquires new cognitive skills by using *internally* a symbolic cognitive tool that was initially used only *externally*. In developmental psychology, the concept of internalization was coined by Vygotsky, who introduced it to explain the development of higher cognitive capacities. Roughly put, according to Vygotsky, higher cognitive functions, such as the ability to read and write, result from the internalization of certain interpersonal practices. A child learning to read is guided by a tutor, who reads to her, shows her the letters, explains how they sound, and helps her read her first words. At the beginning of the learning process, the child may not be able to read longer words by herself, but she is more likely to get through them with the tutor's assistance. For the child, this is the stage at which reading is an interpersonal practice. The learning process is completed when the child becomes able to read by herself, dispensing with the tutor's assistance. At this point, the child has already *internalized* what started as an interpersonal practice. In Vygotsky's words, during the process of internalization

> [a]n operation that initially represents an external activity is reconstructed and begins to occur internally … An interpersonal process is transformed into an intrapersonal one. Every function in the child's cultural development appears twice: first it appears on the social level, and later, on the individual level; first *between* people (*interpsychological*), and then *inside* the child (*intrapsychological*). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All the higher functions originate as actual relations between human individuals (Vygotsky, 1978, p. 56-57; some emphases removed).

Numerical competence is one of these higher functions. Think of a child learning to add and subtract. Initially, the child is introduced to the algorithms of addition and subtraction in external space. The teacher shows how these operations are done with paper and pencil, and guides the child in her first attempts to add and subtract. The child learns by means of this interpersonal process. Later on, the child becomes able to add and subtract with paper and pencil without the teacher's assistance. At this point, what started as an interpersonal process has become an intrapersonal one, and a *first* stage of internalization is concluded. In the case of arithmetical operations, though, there is yet a *second* stage of internalization, which takes place when the child no longer needs paper and pencil for calculations, because she can simulate the operations in her mind (at least in easy cases).

Recently, Vygotsky's conception of internalization has found new developments and interpretations in Menary's conception of enculturation (Menary, 2007a, 2015). Enculturation is the thesis according to which our cognitive capabilities for abstract symbolic thought, such as reading, writing, and mathematics, are due to the acquisition of cultural practices that exploit evolutionarily older neural circuits and transform them through the mechanisms of neuronal recycling (Fabry, 2018; Menary, 2015). The idea is that the enculturation of cognitive tools for symbolic thought *changes* the brain, creating new neuronal structures (i.e., new neuronal networks) that implement new cognitive functions. To offer a rough analogy, it is as if by learning how to skillfully use a saw, a carpenter could end up developing saw teeth on their hands. This will never happen with hands, of course, but internalization/enculturation is possible in the brain because the brain is plastic.

Brain plasticity is best illustrated by responses to brain lesions. Just to mention one among the many studies of the effects of lesions on the reorganization of the brain, there is the case of a boy who had his left cerebral hemisphere removed at the age of two and a half years (Danelli et al., 2013). Before the removal, he was already able to speak. However, as predicted, the removal of the left hemisphere resulted in his losing the ability to speak, since the neural architecture of language is normally located in the left hemisphere. Thanks to brain plasticity, though, after an intensive language rehabilitation program, his language skills recovered within two years, with the right hemisphere taking over the linguistic functions that were originally implemented in the left hemisphere. Over the years, the boy developed near-normal linguistic competence.

This is a striking example of plasticity, but this case also shows that neural plasticity has limits. The areas of the boy's right hemisphere that were recruited for language mirrored the organization that linguistic areas usually take on the left hemisphere. Furthermore, although the boy recovered pretty good linguistic competence, he did not manage to overcome some obstacles, such as difficulties with a few syntactic structures and words. The brain is plastic, but it is not a blank slate: there are some structural factors that constrain neural plasticity. The degree to which neural plasticity is constrained, and by which factors, is a matter of debate, but there seems to be consensus on the idea that there are some functional and organizational principles already encoded in the brain, which we owe to genetic evolution, and which influence how the brain will respond to lesions or other external factors such as learning (Plebe & Mazzone, 2016).

The way enculturation changes the brain is subject to similar constraints. This is reflected by the fact that the neuronal networks that are believed to implement higher cognitive functions acquired through enculturation—such as the abilities to write, read, and calculate—are often localized in the same brain regions, regardless of the educational history of each individual. This regularity is explained by the principle of neural reuse (Anderson, 2010) or neuronal recycling (Dehaene, 2005, 2009b; Dehaene & Cohen, 2007). According to this principle, "neural circuits established for one purpose [can be] exapted (exploited, recycled, redeployed) during evolution or normal development, and be put to different uses, often without losing their original functions" (Anderson, 2010, p. 245). Or, as Dehaene describes it, neuronal recycling is

> the partial or total invasion of a cortical territory initially devoted to a different function, by a cultural invention … Neuronal recycling is also a form of reorientation or

> retraining: it transforms an ancient function, one that evolved for a specific domain in our evolutionary past, into a novel function that is more useful in the present cultural context (Dehaene, 2009b, chapter 3, section 9, para. 9).

The principle of neural recycling/reuse[3] determines which previously existing neural circuits will be re-exploited to support new, learned skills. The criterion is that the new functions will be built upon neural circuits whose original function is close to the new function. For example, the acquisition of reading transforms a cortical area whose original function was related to the recognition of the shapes of objects and human faces. This prior function is re-exploited for the recognition of the shapes of letters and words (Dehaene & Cohen, 2011). As we will see in Chapter 4, the acquisition of numerical competence is believed to recycle neuronal networks originally devoted to the implementation of quantical skills. The process of learning a symbolic system for numbers is believed to "sharpen" neurons localized in these neuronal networks (Dehaene, 2011).

Neuronal recycling also imposes constraints on the invention of cognitive tools. If a cognitive tool is to be internalizable, there must be a brain area responsible for similar functions that will be functionally transformed by the cognitive tool. In other words, culturally created cognitive tools must find their neuronal niche: "[e]ducation-induced changes must fit within the fringe of plasticity left open" by genetic constraints (Dehaene & Cohen, 2011, p. 254). In fact, there is a feedback loop between brain organization and culturally created internalizable cognitive tools. A newly created cognitive tool that fits within the brain's fringe of plasticity provokes transformations in the brain, which give rise to higher cognitive functions and possibly to new cultural inventions, which in turn will lead to new reorganization of the brain, and new cultural inventions, and so on. In this feedback loop between brain and cultural environment, neuronal recycling is an endogenous force that shapes cultural creations, whereas education and training are exogenous forces that drive transformations in the brain. These exogenous forces are exerted by parents, caregivers, teachers, and other tutors who create a learning environment with the kind of stimuli children need to have their brains transformed in the desired way (Menary, 2014).

It is worth contrasting enculturation with folk conceptions of mathematical learning found in the literature on the philosophy of mathematics. Building on findings such as those reviewed in section 2.1, the main tenet of the enculturation thesis in this regard is that mathematical competence in general, and numerical competence in particular, come from the internalization of interpersonal practices mediated by symbolic systems (Menary, 2015). This means that numerical competence is neither innate nor acquired autonomously. To acquire numerical competence, a child needs a structured learning environment where symbolic systems for numbers are already available, and tutors who will guide her through

[3] There are minor differences between the concepts of neuronal recycling and neural reuse as defined by Dehaene & Cohen and Anderson, respectively. For example, the former say that "we use the term 'neuronal recycling' specifically to refer to educational changes that occur in developmental time and without any change in the human genetic make-up" (Dehaene & Cohen, 2011, p. 254), whereas Anderson includes the reuse of neural structures carried out by genetic processes as well, as stated in the quotation above. Given that both expressions refer to very similar processes, I will not distinguish between the two notions here. Jones (2020) claims that the concept of neuronal recycling implies that the recycled brain area loses its previous function, whereas neural reuse does not. However, this is not the case. Dehaene explicitly acknowledges that only parts of the original neuronal network are recycled, so that both the previous and new functions may coexist (Dehaene, 2005).

this environment. This contrasts with the view attributed to Plato according to which we are born with mathematical concepts already encoded in the mind, even though these concepts may not be readily available and must be "remembered" or "intuited." This Platonist view finds resonance in nativist accounts in contemporary cognitive science. According to contemporary nativism, the brain mechanisms responsible for numerical competence are genetically specified and many of our numerical abilities result solely from the maturation of these inborn brain structures (Gallistel & Gelman, 1992). For the nativist, interpersonal practices mediated by symbolic systems for numbers do not engender numerical competence, as the enculturation thesis holds, but are rather an externalization of innate numerical competence. The existence of innate quantical skills gives *prima facie* empirical plausibility to the nativist view. However, as I hinted above and as we will see in more detail in Chapter 3, quantical skills are non-numerical, and therefore the origins of numerical competence are likely to be external, in line with the enculturation thesis.

Another conception of mathematical learning that is in conflict with the enculturation thesis is the oversimplified view held by some philosophers according to which children learn numbers by means of their individual experience with collections or patterns. Two examples are Kitcher (1984), for whom the first insights a child gets into concepts such as set and number come from her experiences with collecting and segregating objects such as "blocks on the floor;" and Shapiro (1997), for whom we acquire knowledge of *ante rem* structures such as the natural-number structure by means of sensory recognition of patterns. As held by the enculturation thesis, though, and as the results from developmental psychology that we will see in Chapter 4 clearly show, children do not learn numbers through direct and autonomous experience with collections or patterns. It is the other way around: to become able to segregate and collect accurately, as well as to recognize numerical patterns larger than three or four elements/repetitions, a child must be taught to count.

In the reverse engineering strategy I am adopting here, the observation that numerical competence results from the enculturation of interpersonal practices mediated by numerals and other symbolic systems suggests that these practices and symbols may be key elements of the reality underlying number concepts. In order words, the suggestion is that number concepts *are about* these practices and symbols. I return to this hypothesis in section 2.5.

2.4 De-semantification and re-semantification

I have been arguing that number concepts and numerical competence in general originate from the process of learning symbolic systems such as the sequence of counting words. If this is so, children learn the symbols *before* they know their meanings, since symbols will become meaningful for them only *after* they have mastered the relevant symbolic systems and related practices. But how is it possible to master a symbolic system without knowing what its symbols mean? And how can symbols whose meanings are unknown give rise to their own meanings? The concepts of de-semantification and re-semantification, as introduced by Krämer (2003) and Dutilh Novaes (2012) respectively, help answer these questions.

The main idea behind de- and re-semantification is the *operational* function of symbols. The denotational and communicational functions of symbols are well known. These two functions presuppose the prior existence of something to be denoted or some mental content to be communicated. When symbols are used to perform operations, though, there is no need to refer to something previously available; symbols can be "de-semanticized" and become mere tokens that are manipulated according to certain rules.

Krämer (2003) introduces the concept of de-semantification in the context of what she calls "operative writing." Some examples of systems of operative writing are the formal languages of logic, the programming languages of computer science, and the language of school arithmetic. Different from other systems of writing, the primary function of systems of operative writing is not the composition of texts for communication, but the solution of problems or the fulfilment of cognitive tasks.

Systems of operative writing are truly cognitive tools. Besides syntactic and semantic rules, these systems also include operational rules.[4] Operational rules specify all or some of the actions that must or can be performed in order to solve a problem. For example, in multiplications with pencil and paper, symbols are written following not only the syntactic rules of the decimal place-value system, but also operational rules codified in multiplication tables and in the multiplication algorithm. These rules determine which symbols must be written at each step and where they must be placed on paper, so that at the end of the process the correct solution is obtained. Other examples of operational rules are the inference rules of formal systems of logic. These rules specify which symbolic transformations are allowed in the system. Formulas that are obtainable exclusively by means of authorized transformations are theorems of the system. Finding a chain of authorized transformations that starts with the axioms and finishes with a target formula is a way of showing that this formula is a theorem of the system. As Krämer puts it, systems of operative writing are at the same time "a medium for representing a realm of cognitive phenomena" and "a tool for operating hands-on with these phenomena in order to solve problems or to prove theories pertaining to this cognitive realm" (Krämer, 2003, p. 522).

A remarkable property of operational rules is that they can be operated mechanically, i.e., without the agent needing to pay attention either to the purpose of the rules or to the meaning of the symbols she is performing the operations with. Usually, both the operational rules and the symbols have intended semantics, but the agent can temporarily "turn off" their semantic content and just make the symbolic transformations prescribed by the rules. This is where de-semantification comes in: when manipulated mechanically, symbols are no longer seen as signs standing for something else, but become self-contained objects, mere links in the chain of steps that are required to fulfil a cognitive task.

De-semantification is a common phenomenon in the execution of arithmetical operations by human agents. As we saw in section 2.1, according to the Triple Code Model, the brain stores de-semanticized arithmetical information in the form of visual representations of Arabic digits (M1) and phonological and graphemic forms of number words as well as rote arithmetical facts (M2). The circumstances in which de-semanticized calculations take place are experimentally identifiable. A number of studies have investigated the strategies people use to mentally solve arithmetical operations (e.g., Caviola, Mammarella, Pastore, and LeFevre (2018); LeFevre et al. (2006); Lemaire and Brun (2017)). These studies have

[4] Operational rules may also be syntactic in the sense that they specify how symbols must or can be combined. However, they go beyond syntactic rules that merely determine how expressions in the language are formed and say nothing about how to operate with them to solve a problem.

identified that the favorite strategy for addition and subtraction of numbers smaller than ten is retrieval of the solution from memory. For instance, when asked to calculate 3+4 and explain how they arrived at the answer, participants in these studies report that they "simply know" that 3+4 is 7. According to the Triple Code Model, in simple operations such as this, people simply retrieve rote, de-semanticized arithmetical facts stored in M1 and M2. If the input is presented in Arabic digits, first the visual form '3+4' is transcoded in verbal format, and then the relevant arithmetical fact is recovered from M2. There is no semantic content involved. De-semantified calculations are even more common on paper. Dehaene illustrates a de-semantified calculation on paper as follows:

> Suppose that you have to compute 24 + 59 … You will have to go carefully through a series of steps: Isolate the rightmost digits (4 and 9), add them up (4 + 9 = 13), write down the 3, carry the 1, isolate the leftmost digits (2 and 5), add them up (2 + 5 = 7), add the carry over (7 + 1 = 8), and finally write down the 8 … At no time during such a calculation does the meaning of the unfolding operations seem to be taken into account. Why did you carry the 1 over to the leftmost column? Perhaps you now realize that this 1 stands for 10 units and that it must therefore land in the tens column. Yet this thought never crossed your mind while you were computing. In order to calculate fast, the brain is forced to ignore the meaning of the computations it performs (Dehaene, 2011, p. 117).

As Dehaene points out, one of the cognitive benefits of de-semantification is that, freed of the cognitive load of taking meaning into account at every step, the brain can operate faster. But de-semantification has yet another cognitive benefit: it is through the mechanical application of operational rules that systems of operative writing can give rise to new contents hardly achievable otherwise. Dutilh Novaes (2012) investigates this point with regard to formal languages in logic. She holds that de-semanticized formalisms have a debiasing effect on reasoning, which enables us to draw conclusions that we could hardly draw with our "bare brains."

Arguing against traditional conceptions of human reasoning, Dutilh Novaes (2012) reviews a number of studies that show that our spontaneous ways of reasoning tend to deviate from the normative canons of logic. These studies show that, if people are not explicitly concerned with following strict logical rules, they are likely to fall prey to "belief bias," i.e., the tendency to accept arguments whose conclusion is in line with one's beliefs and reject arguments whose conclusion confronts one's beliefs, regardless of the validity of the arguments. Under the influence of belief bias, an invalid argument can pass as valid if its conclusion is in line with one's beliefs. Dutilh Novaes maintains that a way of mitigating belief bias is by relying on formal tools. One of the main features of formalisms is the fact that, within formal systems, symbols can be de-semanticized, i.e., manipulated in accordance with the rules of the formal system regardless of any interpretation. When interpretations are laid aside, prior beliefs cannot be invoked, and belief bias is mitigated. Freed from the influence of belief bias, we can achieve a level of rigor in the evaluation of validity that was not possible under its influence.

When not assisted by formal tools, mathematicians and logicians, just like everyone else, are subject to belief bias. This can make the task of proving theorems, especially when it comes to long proofs, very tricky. A sympathy for a tacit principle, or the belief in a yet unproven result, may affect mathematical and logical reasoning and back invalid proofs. This is one reason why formalisms have been so important to promote rigor in contemporary mathematics and logic. Frege seemed to be aware of the debiasing effect of his *Begriffsschrift* when he pointed out that its first purpose was "to provide us with the most reliable test of the validity of a chain of inferences and to point out every presupposition that tries to sneak in unnoticed, so that its origin can be investigated" (Frege, 1967, p. 6). Without having a tool to make explicit all the presuppositions involved in a proof, the evaluation of its validity becomes a much more complex task.

If formalisms are essential to counter belief bias in mathematics and logic—i.e., to prevent previously held beliefs from "sneaking in unnoticed" in mathematical and logical proofs—they can be seen, as Dutilh Novaes does, as indispensable tools for the task of proving mathematical and logical theorems, at least when long and complex proofs are demanded. Such proofs could not be brought about, at least not with the same level of certainty, without the use of formal tools. If this is so, "rather than mere expressions of cognitive processes that take place independently, formal languages are constitutive of these very processes" (Dutilh Novaes, 2012, p. 161). In other words, formal languages do not merely express previously known proofs; they are an indispensable part of the processes through which certain proofs are brought about.

In sum, formal languages are cognitive tools. As unassisted reasoning is not apt for deductions involving long chains of inference, we need to rely on formal techniques. These techniques comprise operational rules that allow us to manipulate de-semanticized symbols mechanically. At the end of the manipulation, we obtain a content (in this case, a proof) that was not previously available. In other words, initially de-semanticized symbols create the content they themselves express.

The same process takes place in arithmetic. We saw in section 2.1 that, prevented from using symbolic resources to treat numerical information, we have to proceed exclusively with our quantical capacities. But our quantical capacities are imprecise: the larger the involved quantities, the larger the errors in estimating and calculating. Equipped only with quantical skills, how could we find the result of 478 + 759? At best, if the input were provided via visual clouds of dots, we could have a very rough, only approximate idea of the result, probably far from the correct result. Unassisted reasoning is not apt for exact calculations involving quantities larger than three or four. To overcome this limitation, we use a symbolic cognitive tool such as the system of operative writing consisting of Arabic digits and the algorithm for addition. The mechanical manipulation of digits according to the operational rules of this system gives us the exact result. At the end of the manipulation, we obtain a content that was not previously available and that is not obtainable without the assistance of a cognitive tool. Again, de-semanticized symbols manipulated according to operational rules create the content they themselves express.

Dutilh Novaes and other philosophers with a cognitivist leaning have observed that systems of numerals and other mathematical symbolic systems are not mere vehicles for the communication of previously available contents, but rather "constitutive" of the very contents they convey. The following are some quotes that capture the gist of the perspective I am endorsing here.

> [E]ven if at specific occasions … 'doing math' does not require the act of manipulating external symbols …, from a diachronic, developmental point of view, external symbols appear to be a necessary condition for the emergence of mathematical concepts and mathematical reasoning (Dutilh Novaes, 2013, p. 55).

> [M]ore ore than language-dependent, exact numerical cognition is external-symbol-dependent; it presupposes the very concept of exact quantities, which may only emerge by means of explicit association to external symbols and the practice of counting beyond very small amounts (Dutilh Novaes, 2013, p. 54).

> Symbols have unique properties that allow for operations—addition, subtraction, multiplication, division, and so on that are much harder (if not unlikely) without them (Menary, 2015, p. 10).

> Symbol systems, such as those for written language and mathematics, are not impermanent scaffolds that we shrug off in adulthood, but are permanent scaffolds that indelibly alter the architecture of cognition (Menary, 2015, p. 20).

> [M]athematical symbols are intimately linked to the concepts they represent … symbols are not merely used to express mathematical concepts, but … they are constitutive of the concepts themselves. Mathematical symbols enable us to perform mathematical operations that we would not be able to do in the mind alone, they are epistemic actions (De Cruz & De Smedt, 2013, p. 4).

> [A]n advanced understanding of arithmetic, which includes performing number comparisons and mental arithmetic, is mediated by symbolic reasoning with external representations. … [E]xternal representations are not just some arbitrary way of expressing thoughts, but they are constitutive of our thoughts themselves (Schlimm, 2018, pp. 196, 208).

It is hardly controversial that, from the perspective of an individual, the symbolic cognitive tools of arithmetic are indispensable for the acquisition of contents that the individual herself could not obtain by other means. However, whereas it is impossible for me to obtain by myself the result of adding 478 to 759 without operating with symbols (either mentally or with pencil and paper), the result of this addition is not an original idea at the historical level in any sense. I personally may never have entertained this particular addition, but the symbolic system I use and its semantics were already in place many centuries before I was born. Thus, even if the decimal place-value system of number notation is indispensable for my arithmetical concepts, this does not imply that it is indispensable for the emergence of numerical concepts in historical terms. The decimal place-value system was created at some time in the past by people who certainly already knew numbers and arithmetic operations. These people probably used a more ancient notation system (Yong & Se, 2004), which gave them numerical competence. But if we ask about the origins of this more ancient numeral system, this can easily lead to an infinite regress. At some point, it seems, there should be someone who invented the very first notation system without having acquired numerical competence from experience with any symbolic system for numbers (Pelland, 2018a). But how was this possible, if numerical competence, as we saw in section 2.1, is unavoidably symbolic?

Dutilh Novaes's (2012) concept of *re-semantification* can help answer this question. Re-semantification refers to the action of giving a formalism a semantic interpretation that was not the one intended when the formalism was first developed. Simply put, "[t]he idea behind re-semantification is that a formalism which is developed to characterize a specific phenomenon *A* can then be reinterpreted on another phenomenon *B*" (Dutilh Novaes, 2012, p. 204). Reinterpretation of formal systems is a common practice in mathematics and logic, especially in model theory, where the many models that a given axiomatic system can have besides its intended one (if any) is a topic of investigation. But the cases of re-semantification that I am interested in here are significantly different from mere cases of reinterpretation, in that it is the *de*-semantification of the formalism under consideration and the mechanical computations that ensue, that make room for the creation of a new, original model.

Dutilh Novaes illustrates how re-semantification can give rise to novelties through an example from the history of physics: the development of Maxwell's theory of electromagnetism. Maxwell's initial goal was to give a mathematical formulation of Faraday's account of electromagnetism. As Dutilh Novaes points out, his first two mathematical models included the hypothesis that electromagnetic waves propagated through ether. However, when Maxwell was elaborating his third and final mathematical formulation of electromagnetic waves, he realized that the mathematical formalism could be made simpler by excluding from it the assumption about the existence of a medium through which waves propagated. "Maxwell let 'the mathematics speak for itself' … He treated the mathematical formalism as 'de-semanticized' rather than letting himself be guided by his own preferred interpretation" (Dutilh Novaes, 2012, p. 214-215). By doing so, he made his model neutral regarding the existence of ether, even if he himself believed in its existence. This episode illustrates the power of formalisms to counter belief bias: the formalism suggested, against Maxwell's own beliefs, that the ether hypothesis was dispensable. Maxwell's de-semantification of the formalism ended up suggesting a surprising original interpretation of it. This was the re-semantification step, whereby a new "model"—one wherein ether does not exist—was created.

In this example, the difference between the two models can be regarded as small, even if extremely significant and with striking consequences. Furthermore, it does not seem that Maxwell's formalisms were indispensable, in any sense, for the emergence of the hypothesis that electromagnetic waves can propagate without a medium. Sooner or later this fact might have been discovered by other means. Even so, this case shows the potential of re-semantification to give rise to original contents.

Re-semantification (in cases like this) is a process of de-semantification in which the mechanical operation of a formalism ends up giving rise to a new model for the same formalism. When re-semantification takes place, the symbols that were originally used to refer to a domain are reinterpreted as referring to another domain, which was not available previously but was instead suggested by the use of the symbolic system itself. This may explain how the very first numeral systems may have emerged in the absence of anyone with numerical competence: it is sufficient that there was a symbolic system already in use to deal with another domain that, through re-semantification, gave rise to a numerical domain. Making this possibility historically plausible demands a whole chapter. I develop this proposal in Chapter 5.

Now we are in a position to answer the two questions posed at the beginning of this section. The first was: how is it possible to master a symbolic system without knowing what its symbols mean? From the perspective of one individual, in systems of operative writing this is possible if one masters the operational rules first. By manipulating symbols mechanically, one acquires experience with them and then becomes able to appreciate the contents that these manipulations deliver. The details of how this happens during the acquisition of numerical competence are addressed in Chapter 4. The second was: how can symbols give rise to their own meanings? In historical terms, re-semantification is the process that explains how symbols initially applied in one context can give rise to their own new meanings in a new context.

If de- and re-semantification can answer these questions, then the hypothesis according to which numerical competence results from the internalization of symbolic cognitive tools becomes more plausible, and so does the hypothesis according to which number concepts are about these cognitive tools. I elaborate on these hypotheses in the next section.

2.5 The hypotheses

What does all this discussion about cognition and cognitive tools tell us about the existence of numbers? In order to tie things together, I first need to distinguish three different but interrelated questions about numbers that I am addressing in this dissertation. First, there is the *cognitive question*, which asks for a description of the cognitive processes underlying human numerical competence. This is the question I have focused on in the previous sections. Second, there is the *epistemic question*, which asks whether numerical competence amounts to some kind of propositional knowledge. Surely, numerical competence is a kind of know-how, but in the epistemic question we are concerned with determining whether a statement such as '3+4=7' is true and, if so, true of what. This leads us to the third and, here, most important question—the *ontological question*—which asks what exists in the domain of arithmetic.

The results and theories briefly reviewed above suggest an answer to the cognitive question according to which numerical competence originates from the internalization of external symbolic cognitive tools and associated practices, with some contributions from quantical cognition. In the reverse engineering strategy I proposed in the introduction to this chapter, this suggests answers to both the epistemic and ontological questions.

As for the epistemic question, the suggestion is that propositional knowledge in arithmetic is nothing more than knowledge of descriptions of the workings of the very same cognitive tools that give rise to numerical competence. In the least controversial cases of knowledge, the beliefs we label as 'knowledge' come from direct experiences with particular objects or situations. For example, I can claim that I know that my father's dog bubbles with excitement when she sees her walking harness because I have seen her do this daily. Knowledge claims such as this hinge on the causal theory of knowledge, so often criticized in attempts to solve Benacerraf's problem. But, if one assumes a causal theory of knowledge in the case of arithmetic and remains neutral about the existence and nature of numbers (as my reverse engineering strategy recommends), the fact that numerical competence originates from mastering certain cognitive tools naturally suggests that arithmetical statements may be true of these very same cognitive tools and practices. In other words, the suggestion is that arithmetical statements may be seen as *describing* features of these cognitive tools and practices. A simple example helps illustrate this point. According to this suggestion, the statement '3+4=7' describes what happens if one starts counting at 'three' and moves forward four positions in the counting sequence. Thus, '3+4=7' would be true of this very operation. I develop the details of this account in Chapter 7.

This possible answer to the epistemic question naturally suggests an answer to the ontological question. Insofar as arithmetical statements are seen as describing features of certain cognitive tools and practices, these cognitive tools and practices can be seen as the reality underlying arithmetic. In this view, rather than being about a realm of non-spatiotemporal objects, arithmetic is about the cognitive tools and practices that give rise to numerical competence. For example, the objective reality underlying '3+4=7' is no longer seen as a realm of abstract objects, but rather as the counting procedure conceived of as a human practice (more on this in Chapter 7).

There are many loose ends in the answers to the cognitive, epistemic, and ontological questions briefly sketched above. At this point, they are to be taken as hypotheses. These hypotheses will guide the investigation in the remaining chapters of this dissertation.

Most of the loose ends in my answer to the cognitive question are addressed in the next three chapters. One of the main points in need of further investigation is the non-numerical nature of quantical cognition. I address this point in Chapter 3. Another key point in need of further support is the claim that we acquire numerical competence by experiencing numerals initially as de-semanticized symbols governed by operational rules. In Chapter 4 I review findings from developmental psychology and numerical cognition that provide empirical support for this claim. A third point that deserves further investigation is the historical origins of the first numeral systems and counting procedures in the absence of anyone with numerical competence. I address this point in Chapter 5.

A difficulty that I have not mentioned so far concerns the consequences for the syntax and semantics of arithmetical statements when replacing an ontology of objects with an ontology of cognitive tools and human practices. How can a statement such as '3+4=7' be true of symbolic systems and procedures if its literal reading seems to refer to objects? My answer to this relies on Sfard's (2008) account of mathematical learning, according to which, during the process of learning arithmetical operations, we need to *reify* initial segments of the counting procedure so as to make operations with larger numbers cognitively easier. According to Sfard, children start learning operations such as addition and multiplication as higher-order counting procedures. For example, to add four to three, a beginner will count four fingers, then count three fingers, and then count all of them in order to obtain the final result. This strategy quickly becomes cumbersome as the involved numbers grow. Thus, to succeed in operations involving larger numbers, the child has to *encapsulate* initial segments of the counting procedure into discursive objects so that she can operate directly with these discursive objects, thus reducing the complexity of the operations she has to perform. By doing so, numerical statements that were initially experienced as commands to perform successive counting operations start to be seen as statements referring to numbers—i.e., discursive objects produced by the reification of segments of the counting procedure. This explains why numerical terms are seen, and should be seen, as singular terms. The referents of these singular terms—discursive objects produced by reification—function as truly cognitive tools that make complex arithmetical operations cognitively easier. The various loose ends of this story are addressed in Chapter 6 and section 7.1.

Finally, the role of reification in mathematical learning suggests a fourth hypothesis: numbers as platonic entities do not exist; what is really important is the *thought* that numbers exist (i.e., the process of reification). Once numbers—the referents of numerical singular terms—are explained away as useful reifications, we have an account of arithmetic in which arithmetical statements describe properties of certain cognitive tools and there are no existing numbers. In Chapter 7 I explain how this nominalistic hypothesis allows for an explanation of the epistemic properties of arithmetic such as truth and objectivity, making superfluous the postulation of non-spatiotemporal numbers.

2.6 Conclusion

In this chapter, I started reviewing some findings from numerical cognition that track the origins of numerical competence to symbolic cognitive tools. I also introduced concepts and theories from the psychological and philosophical literature on cognitive tools that will inform the discussion in the next chapters. Finally, based on these results and theories, I introduced a hypothesis about the nature of numbers according to which numbers are cognitively useful reifications, and arithmetical knowledge is, in fact, knowledge about the suite of techniques that make up the cognitive tools of arithmetic.

Chapter 3

Quantical cognition

A KEY element of the hypotheses suggested in section 2.5 is the observation that numerical competence originates from the internalization of external symbolic cognitive tools such as the counting procedure. If this is so, numerical competence is not innate. However, we saw in section 2.1 that one of the most remarkable findings from numerical cognition is the fact that human infants and non-human animals share an inborn set of abilities to identify and discriminate between discrete quantities. What these infants and non-human animals are able to do, we, numerate adults, usually do by relying on numerical symbols. For example, when we subitize or estimate the cardinal size of a collection, we express the outcome by uttering a numeral ("three") or a numerical expression ("about ten"). Infants and non-human animals cannot express the outcome of their assessments of cardinal size in this way, since they do not know culturally-created numerals, but it may be that they rely on some innate, *non-symbolic* kind of numerical competence. This is what the nativist hypothesis on numerical cognition holds. This hypothesis has been supported by influential psychologists such as Gallistel and Gelman (1992) and Butterworth (1999).

On the other hand, the externalist hypothesis—according to which truly numerical competence originates from familiarity with external symbolic systems—also has strong advocates, such as Carey (2009) and, to some extent, Dehaene (2011). Following Núñez (2017), I already anticipated in section 2.1 that the abilities to subitize and estimate are best seen as non-numerical. I called these inborn non-symbolic abilities *quantical cognition*. In this chapter, I substantiate this claim.

There are two possible lines of argumentation for the view that non-symbolic numerical cognition is non-numerical. Some cognitive scientists have argued that non-symbolic numerical cognition is not really numerical because the abilities to subitize and estimate may be guided by non-numerical cues whose variation is hardly distinguishable from cardinal size, such as surface area, density, and total amount of matter (Leibovich, Katzin, Harel, & Henik, 2017). For example, it may be that in order to distinguish a collection of two from a collection of three puppets, infants observe the total amount of matter, which varies with number, rather than the number of puppets. To tackle this issue, almost all recent studies include controls for number-correlated magnitudes so that their effect on the observed results can be isolated. However, these measures have not yet been sufficient to settle the debate

(Henik, 2016). This is not the line of argumentation I will follow in this chapter. My point is that, even if this debate is eventually settled in favor of the hypothesis that subjects are really guided by cardinal size, the implementation of subitizing and estimation in the brain is unlikely to rely on any numerical resource, as the very models advanced by the cognitive scientists who view these abilities as numerical show. The misattribution of numerical competence to subjects endowed with quantical abilities comes from the loose use of technical terminology already noted by Núñez (2017). I will show that, once three key concepts are carefully distinguished—numerosity, cardinality, and number—it becomes clear that quantical abilities can track numerosities even if they do not rely on numerical resources.

This chapter is structured as follows. In section 3.1, I describe the different kinds of quantical skills that humans share with non-human animals. I pay special attention to the disparity between the outcomes produced by quantical abilities and those that should be expected from symbolic counting and arithmetic. This will be useful not only for the discussion in this chapter—this disparity is one of the main reasons why number concepts must not be involved in quantical cognition—but also for the following chapters, where I show how the introduction of symbols (numerals) enables us to overcome the limitations of our inborn quantical skills and obtain number concepts. In section 3.2, I provide an outline of the cognitive models intended to explain the mechanisms underlying quantical abilities, namely, the Object File System (OFS) and the so-called "Approximate Number System" (ANS). In section 3.3, I briefly present neuroscientific findings that have shed light on where and how the OFS and the ANS are implemented in the brain. In section 3.4, I outline the evolutionary origins of quantical cognition. Section 3.5 is where I argue that quantical cognition does not involve numbers in any sense. Finally, in section 3.6 I bring to light what the discussion in this chapter tells us about the nature of numbers: numbers are neither a perceivable property of the environment nor a genetically evolved cognitive resource. This conclusion reinforces the view—to be defended in Chapters 4 and 5—that numerical cognition results from the internalization of culturally-created cognitive tools.

3.1 Quantical abilities

The literature on numerical cognition distinguishes between two broad categories of abilities: the ability to *calculate*, i.e., the ability to perform operations such as addition and multiplication, and the ability to *enumerate*, i.e., the ability to assess the cardinal size of a collection. Both calculation and enumeration can involve symbolic manipulation or not. In this chapter, I focus exclusively on the non-symbolic manifestations of these abilities. As anticipated in the introduction, in section 3.5 I will argue that non-symbolic calculation and non-symbolic enumeration do not involve numbers, and so it seems inadequate to call these abilities "calculation" and "enumeration." However, for lack of more suitable terms in the literature, I will use the conventional terminology.

Many of the studies I consider next were conducted with infants or non-human animals, which completely lack number words and cannot rely on other culturally-created symbolic resources. In studies with numerate human adults aimed at identifying quantical abilities, experimenters usually make use of techniques that prevent participants from using linguistic resources, such as presenting test stimuli too briefly to be counted. When reviewing the relevant literature in the following sections, I will not mention the multiple strategies experimenters employed to prevent the use of symbols, nor the controls they introduced to isolate the effect of numerical variation from the effect of co-occurring variations in continuous magnitudes. But keep in mind that these controls are present in almost all studies cited.

3.1.1 Non-symbolic enumeration

Enumeration (or quantification, such as in Dehaene (1992)) is the ability to determine the numerical size of a collection. Counting is the most common symbolic method of enumeration for numerate humans, but it is by no means a synonym of enumeration, at least when it comes to the way these terms are used in the field of numerical cognition. In order to count, we need to establish a one-to-one correspondence between the items to be counted and a initial segment of the sequence of counting words. This process demands several smaller operations, such as shifting the attentional focus from one item to the next, keeping track of which objects have already been counted, and so on. However, there are situations where we are able to grasp the number of objects in a collection without needing to go through these serial steps which characterize counting. We just seem to instantaneously grasp the total amount of items at a glance, even if only approximately on certain occasions. Despite the fact that we can express the result of such an operation by reporting a numeral, we do not explicitly count in these "at a glance" cases. The term 'enumeration' encompasses the multiple operations through which we can grasp or determine the size of collections, whether involving symbols or not; counting is just one of these methods.[1]

The property of collections related to their cardinal size that we can grasp at a glance by non-symbolic means is usually called *numerosity*. I will give a precise definition of numerosity in section 3.5. For the time being, we can understand numerosity as the perceivable cardinality of a collection of objects, such as an array of dots or a sequence of tones, the kinds of stimuli most commonly used in the experiments I cite in this section. In terms of the concept of numerosity, non-symbolic enumeration may be defined as the ability to grasp the numerosity of a perceivable collection without explicitly counting its items.

At least two modes of non-symbolic enumeration have been reported. The most paradigmatic one is *subitizing*. We are able to quickly and accurately "see" the numerosity of small collections comprising up to three or four objects, without having to count them. Although we cannot subitize larger collections, when prevented from counting we can still *estimate* their sizes. Estimation is much less accurate than subitizing, but it is very far from being a wild guess, as we will see. Let us examine subitizing and estimation in turn.

Subitizing Subitizing is defined as the fast and accurate enumeration of collections of up to three or four elements without active verbal counting.[2] Kaufman, Lord, Reese, and Volk-

[1]Some authors do not follow this distinction. For example, Dacke and Srinivasan (2008) and Skorupski, MaBouDi, Dona, and Chittka (2017) speak of "counting insects," although they do not intend to claim that insects are really able to engage in symbolic counting.

[2]Most of the terms I present definitions of here do not have widely accepted standard definitions in the literature. For example, Nieder (2016, p. 366) defines subitizing in terms of the mechanism that is believed to implement it: "[s]ubitizing (also known as object file representation or object tracking system). The rapid

mann (1949) proposed the neologism 'subitizing' to name this phenomenon so as to highlight the apparent immediateness of enumeration. In Latin, the verb *subitare* means to arrive suddenly. However, subitizing is not really instantaneous. In numerate adults, the enumeration of a visual stimulus comprising one dot takes about 400 ms—the lapse between presentation of the stimulus and reporting of the numeral. For each added dot, the enumeration time increases by about 40 to 100 ms (Trick & Pylyshyn, 1994). Besides being fast, subitizing is errorless (Kaufman et al., 1949; Mandler & Shebo, 1982). These are its most remarkable characteristics.

Gallistel and Gelman (1991) defend the view that subitizing is no more than a kind of fast, non-symbolic counting, carried out by a mental mechanism that uses non-verbal tokens instead of counting words. Regardless of the internal mechanism underlying subitizing being similar to counting or not, subitizing is clearly distinguishable from active verbal counting in terms of reaction time, since verbal counting is much slower. Figure 3.1 plots mean values of reaction time for the enumeration of arrays of dots ranging from one to eight items. As can be seen in the graph, the relationship between reaction time and numerosity is described by a bilinear function, in which the subitizing slope is less steep than the counting slope. The discontinuity in the slope at four items indicates the change in the enumeration regimen from subitizing to active verbal counting.

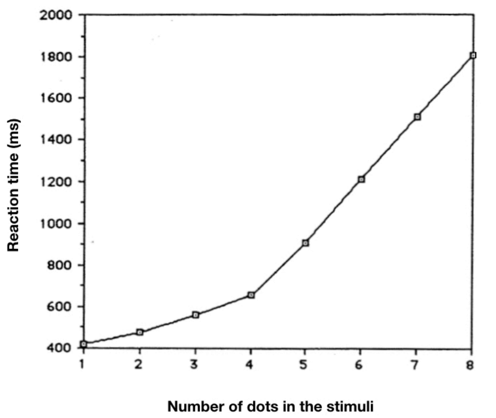

Figure 3.1: The graph plots the reaction time of numerate people asked to say how many dots there are in a given stimulus as fast as they can. Subitizing takes 40-100 ms/item, whereas counting takes 250-350 ms/item. (Figure adapted from Trick and Pylyshyn (1994, p. 81).)

Subitizing was first detected in numerate adults, who can verbally report the number of elements they see. The detection of subitizing in infants, young children, and non-human animals demands different experimental setups, since they cannot do the same. Because subitizing requires the subject to *report* a numeral, there has been some debate about whether it is appropriate to speak of subitizing in non-human animals (see Agrillo (2015); Davis and Pérusse (1988); Miller (1993)). Notwithstanding this debate, the point I want to highlight is that infants and animals are able to accurately distinguish numerosities within the subitizing range, despite the fact that they cannot verbally report it. There is plenty of evidence for this in the literature. I will mention just a small sample of it.

Most of the experiments with infants use the method of habituation/dishabituation, also known as violation of expectations. In this kind of experiment, infants are first habituated to a stimulus; for example, a certain number of dots is presented repeatedly. As the infant becomes habituated, looking time de-

tracking for up to approximately four items by assigning 'files' or 'pointers' to individual items." I will adopt definitions which characterize quantical abilities exclusively in terms of their behavioral features, independently of the underlying processes proposed to explain them.

creases. Then, a different number of dots is presented. If the infant looks longer at the new stimulus, it is taken as a sign that she recognized the difference in the number of dots. Starkey and Cooper (1980) used this method to probe the abilities of 22-week-old human infants to distinguish small numerosities. They habituated infants to a stimulus containing two dots, and then presented a stimulus with three dots; infants looked longer at the new stimulus. Antell and Keating (1983) replicated the same result with 53-hour-old neonates. Using a different experimental setup which allowed for more reliable control of non-numerical variables, Starkey and Cooper (1995) tested 2-year-old children who had not yet mastered number words and counting in a numerosity comparison task. Presented with a pair of stimulus arrays for only 200 ms, children were asked to judge whether the stimuli were "the same" or "not the same" regarding numerosity. Children exhibited high accuracy in the comparisons between numerosities within the subitizing range. Results such as these have consistently demonstrated that subitizing is in place long before children learn to count.

Another indication that the ability to subitize is innate is that non-human animals also exhibit accurate perception of numerosity within the subitizing range. In an experiment conducted with rhesus monkeys, Hauser, Carey, and Hauser (2000) showed that, when confronted with two containers with different quantities of apple slices, monkeys successfully choose the greater quantity if both quantities are within the subitizing range, but fail more often when at least one collection is above four. Hunt, Low, and Burns (2008) showed that the New Zealand robin exhibits the same ability when confronted with different numbers of mealworms. More human-like subitizing has been observed in chimpanzees trained to use numerals. Murofushi (1997) trained a female chimpanzee to match arrays of dots with Arabic numerals representing their numerosity. The chimpanzee was able to label arrays with up to seven dots quite accurately. Reaction time and error rate suggested that her performance was based on subitizing for up to three dots. For four or more dots, her performance was consistent with estimation.

The ability to accurately enumerate small collections of objects has also been observed in insects. Dacke and Srinivasan (2008) showed that bees are able to keep track of the position of a food reward by enumerating the landmarks they passed, provided that the number of prominent landmarks does not exceed four. Non-symbolic enumeration has also been observed in other invertebrate species, such as cuttlefish, ants, and spiders (Skorupski et al., 2017).

Estimation Above the limit of subitizing, when prevented from or unable to count, subjects estimate. Estimation is defined as the ability to produce an approximate appraisal of the number of items in a collection, without active verbal counting. In contrast to the outcomes of subitizing, estimates are highly imprecise and vary greatly across individuals. Minturn and Reese (1951) asked participants to assess the number of dots in a display. Figure 3.2 shows the dispersion of their responses. Up to five dots, responses coincide perfectly among participants and are completely accurate. In this interval, participants likely used subitizing or a combination of subitizing and counting. Above five, however, the variability in the median responses increased steadily. For a display with 200 dots, for example, estimates ranged from 50 to 700. These results exemplify how inaccurate estimates are.

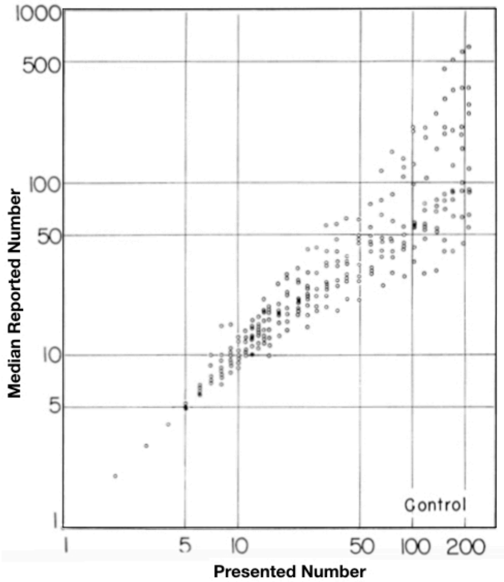

Figure 3.2: Dispersion of estimates. Minturn and Reese (1951) tested 10 participants. Each point represents the median estimate of a participant over five trials. (Figure adapted from Minturn and Reese (1951, p. 209).)

However, the data depicted in Figure 3.2 also shows that estimates are far from being random. They follow a regular pattern: the dispersion of the responses increases in direct proportion to the presented number of items. It is as if there is an increasing occurrence of "noise" in the perception of numerosity. The larger the numerosity, the larger the noise, and therefore the larger the dispersion of responses across individuals. The statistical pattern that governs this increment in the dispersion of estimates was dubbed *scalar variability* by Whalen, Gallistel, and Gelman (1999). The so-called "law of scalar variability" states that the mean estimate of the number of items in a given collection and its standard deviation[3] increase proportionally to each other as the number of items increases, so that the coefficient of variation is constant across the presented number of items.[4] Figure 3.3 depicts this effect for the seven participants of Whalen's et al. (1999) experiment.

This regular increase in the dispersion of estimates is thought to be caused by increasing imprecision in the subjects' internal representation of numerosity, as we will see in the next section. For now, what is important to highlight is that, although the ability to estimate is highly imprecise, it produces consistent estimates—i.e., the mean response increases with the number of presented items—and these estimates vary within a proportional range of dispersion. Scalar variability is just one among other regularities that can be observed in estimation noise. The perception of numerosities above subitizing range is also subject to the influence of two other psychophysical effects, namely, Weber's law and sensory adaptation. These effects also affect the perception of many other physical magnitudes. Let us consider each in turn.

In loose terms, Weber's law states that our capacity to discriminate two stimuli of different intensities is inversely proportional to their ratio. For example, when it comes to our perception of weight, it is easier to discriminate a one-kilo object from a two-kilo object (ratio 1:2 or 0.5) than a three-kilo object from a four-kilo object (ratio 3:4 or 0.75). In more precise terms, Weber's law is defined in terms of the concept of just noticeable difference (*JND*), also known as the discrimination threshold. The value of the *JND* for a given sensible magnitude is experimentally determined as the smallest variation in the intensity

[3]The standard deviation is a measure of the dispersion of the responses around the mean.

[4]The coefficient of variation (cv) is calculated as the ratio of the standard deviation (sd) to the mean (m): $cv = sd/m$. It measures the extent of variability in relation to the mean.

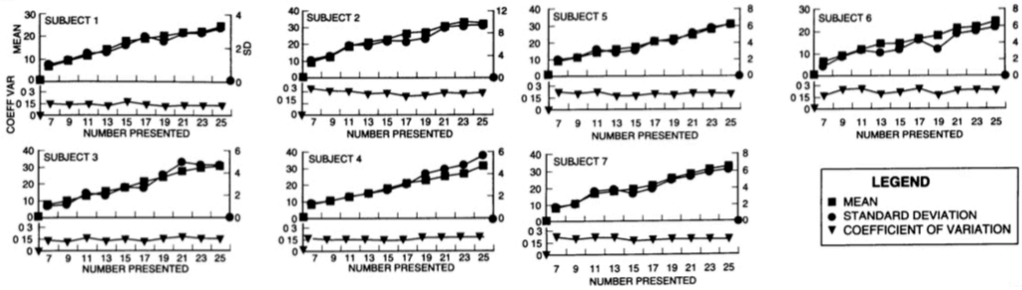

Figure 3.3: In the experiment reported in Whalen et al. (1999), numerate human adults were asked to press a key a predetermined number of times as fast as they could, without counting. This required participants to estimate the number of times they pressed the key. The mean responses of each participant over 40 trials as a function of the target number of pressings and their standard deviations are depicted in the graphs above. Notice that means and standard deviations were plotted in different scales to emphasize that they increase proportionally to each other. The almost constant values of the coefficient of variation demonstrate that the estimates of all participants presented scalar variability. (Figure adapted from Whalen et al. (1999, p. 133).)

of the stimulus that is noticeable to the tested subjects above chance level. For example, the *JND* for the perception of a one-kilo weight will be the smallest increase or decrease in weight that subjects can notice. Weber's law, then, states that the ratio between *JND* and the intensity of the original stimulus is constant. In a formula: $K = JND/I$, where *K* is a constant, known as the Weber fraction, and *I* is the intensity of the original stimulus. As a result, $JND = K.I$, which means that the just noticeable difference in the intensity of a magnitude increases linearly as a function of the intensity of the original stimulus (*I*) and the Weber fraction (*K*). Then, once the Weber fraction for a sensible magnitude is determined, the threshold of discrimination can be predicted for any other intensity of this kind of stimulus in the mid-range levels of stimulation.[5]

When it comes to numerosity, the intensity of stimulation is measured in cardinal numbers. For numerate adult humans, Oeffelen and Vos (1982) found a Weber fraction of 0.162. This means that for a stimulus of intensity, say, 20, the threshold of discrimination will be 3.24. Therefore, an array of 20 dots will be almost indistinguishable from an array of 22 dots when subjects are prevented from counting, because the difference between them—two—is bellow the threshold of discrimination—3.24. When an array of 20 dots is compared to an array of 24 dots, though, subjects are likely to notice the difference above chance level, since the difference—four—is above the threshold of discrimination.

The effects of Weber's law on numerosity discrimination are often presented as comprising two aspects, dubbed *distance* and *size* effects (Dehaene, Dehaene-Lambertz, & Cohen, 1998). The distance effect refers to the fact that the discrimination of numerosities that are numerically distant from each other is easier and more accurate. The size effect refers to the fact that, when numerical distance is kept constant, pairs of smaller numerosities are more easily discriminated from each other. Some examples: the distance effect is observed in the

[5] Weber's law breaks down for very low- or very high-intensity stimuli. For an overview of the discussion about Weber's law in psychology, see Laming (2008).

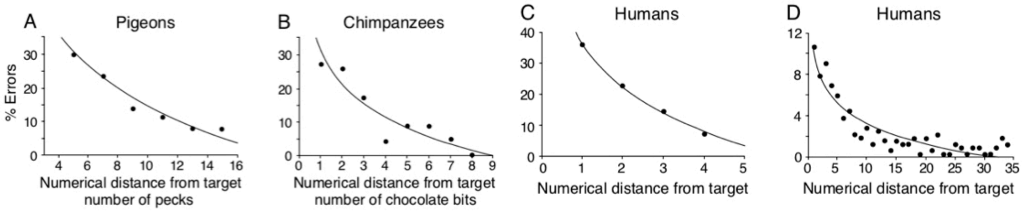

Figure 3.4: Distance effect in numerosity discrimination as observed in pigeons (A), chimpanzees (B), and humans (C, D). In humans, the distance effect appears not only in the discrimination of non-symbolic numerosities, as depicted in (C), but also in the discrimination of Arabic numerals, as depicted in (D). In all species, error rates decrease as numerical distance increases. (Figure adapted from Dehaene et al. (1998, p. 358).)

fact that it is easier to discriminate an array of 20 dots from an array of 10 dots—a numerical distance of 10—than an array of 20 dots from an array of 18 dots—a numerical distance of two. When the numerical distance is kept constant, the size effect is observed in the fact that it is easier to discriminate an array of 20 dots from an array of 10 dots, than an array of 90 dots from an array of 100 dots. Figure 3.4 illustrates the distance effect as observed in humans and non-human animals.

Sensory adaptation, another psychophysical effect affecting numerosity perception, is defined as the adjusted—diminished or increased—sensitivity to a stimulus as a result of constant exposure to that stimulus (Sutherland, 1989, p. 9). Like Weber's law, adaptation also affects many sensible properties, such as the perception of light, smell, and sound (Roeckelein, 2006, p. 8). Adaptation is seen, for example, when after long exposure to the roar of a running air conditioner, we no longer hear it with the same intensity; we adapt to the noise and our sensitivity diminishes. By contrast, when we are in a very silent place, we can hear very low sounds such as the one produced by our heart beating; we adapt to silence and our sensitivity increases.

Burr and Ross (2008) showed how the perception of numerosity is also susceptible to adaptation. After 30 seconds of exposure to a large numerosity (without receiving any information about how many items it contained), participants in their experiment tended to perceive subsequent numerosities as smaller than they did in the control experiment where no adaptation took place. In the opposite direction, after adaptation to a small numerosity, participants tended to perceive subsequent numerosities as greater in comparison to the control experiment. As a rule, estimates become smaller after adaptation to a large numerosity and become larger after adaptation to a small numerosity, resulting in a tendency to underestimate and overestimate respectively (see Figure 3.5). Aagten-Murphy and Burr (2016) demonstrated that adaptation to visual numerosity is spatially specific, i.e., different regions of the visual field can be adapted to high, low or neutral stimuli, and this can occur after only very brief exposure to adapting stimuli. With only one second of exposure, they obtained almost the same adaptation effect that Burr and Ross (2008) had obtained with 30 seconds of exposure.

The distinct behavioral characteristics of estimation and subitizing suggest that they are produced by different underlying cognitive processes. The fact that estimation is subject to the same psychophysical laws that govern perception in general has led many to suggest that humans and non-human animals share a *number sense*, i.e., a sensory system dedicated to the perception of "number" (e.g., Dehaene (2011) and Burr, Anobile, and Arrighi (2017)). Not being subject to the same laws, subitizing is thought to result from a different underlying mechanism. Before addressing the mechanisms underlying non-symbolic enumeration, though, let us take a look at another kind of quantical ability.

3.1.2 Non-symbolic calculation

In addition to the ability to subitize and estimate, human adults, human infants and non-human animals also share the ability to perform simple non-symbolic operations analogous to arithmetic operations. In arithmetic, *symbolic* calculations are usually seen as operations performed with numbers. In numerical cognition studies, however, calculation is most often seen as operations performed by an agent with physical objects with a practical purpose. In this manner, Dehaene (1992, p. 6) defines symbolic calculation as "the ability … to predict by symbolic manipulation the result of a physical regrouping or partitioning act without having to execute it." Calculation is prediction: an agent who is able to calculate is able to anticipate that three items added to five items will make eight items without needing to see all items together.

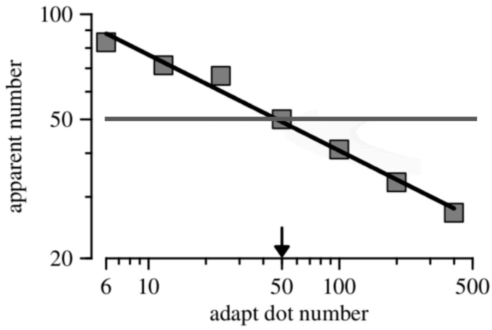

Figure 3.5: Apparent numerosity of an array of 50 dots as a function of different adapting stimuli, ranging from 6 to 500 dots. The perceived numerosity of the 50-dot array increased after adaptation to lower numerosities, decreased after adaptation to higher numerosities, and was not affected after adaptation to the same numerosity. (Figure adapted from Burr et al. (2017, p. 2).)

An explicit definition of non-symbolic calculation is rarely given in the literature, but we can easily obtain one by appropriately modifying Dehaene's definition of symbolic calculation. Thus, *non-symbolic calculation* may be defined as the ability to predict the numerosity resulting from a physical regrouping or partitioning act without having to execute it (nor to see it being executed) and without using any symbolic resources.

The *locus classicus* for studies in non-symbolic calculation in infants is Wynn (1992a). Using violation-of-expectation experiments, she showed that five-month-old human infants can predict the outcome of analogues of the arithmetical operations 1+1 and 2-1 performed with physical objects. In the experimental setting she used to probe infants ability to "calculate" 1+1, infants were presented with one doll, which was subsequently occluded by a panel. Then, infants saw the experimenter placing a second doll behind the occluder. When the panel was removed, infants could see either one or two dolls. Wynn observed that infants looked longer at the wrong outcome—1+1=1—, suggesting that they had formed the expectation of seeing two dolls. Similarly, in the scenario to test 2-1, infants were presented with two dolls, which were subsequently occluded by a panel, and then they saw the experimenter removing one doll from behind the occluder. Again, infants looked longer at the incorrect outcome—2–1=2—suggesting that they had predicted the correct result. Subsequent experiments were conducted to confirm that infants respond to the number of items, rather than to other features of the stimuli. For example, McCrink and Wynn (2004, 2009) used a similar violation-of-expectation experiment, but they presented stimuli in video format, allowing them to control for variables correlated with number, such as area and contour length of the presented objects, so that a possible influence of these variables could be ruled out. With these controls in place, they showed that 9-month-old infants form expectations about the outcomes of physical analogues of the operations 5+5, 10-5, 6+4, and 14-4. This shows that infants' ability to calculate goes even beyond the subitizing range.

Unsurprisingly, non-symbolic calculation involving numerosities above the subitizing range is only approximate. In Barth's et al. (2006) experiment I already mentioned in section 2.1, the performance of participants in non-symbolic calculation was subject to Weber's law. In one experimental setting, they presented participants with three arrays of dots and asked whether the third array had more or fewer dots than the sum of the first two arrays. Correct answers increased as the ratio between the actual result and the proposed outcome increased, in line with distance and size effects.

Noise due to estimation is not the only phenomenon that affects non-symbolic calculation. McCrink, Dehaene, and Dehaene-Lambertz (2007) showed that non-symbolic calculation is biased by what they call *operational momentum*. They presented adults with hundreds of short videos where collections of objects were added or subtracted from one another, and asked whether the final number of items was correct or not. Participants' responses displayed a systematic bias toward larger values in addition and smaller values in subtraction. In other words, answers for addition problems were overestimated, whereas answers for subtraction problems were underestimated. For example, for the operation 8+8, subjects tended to consider the displayed outcome as correct when it had about 20 items, thus overestimating the result of the non-symbolic operation. By contrast, for the operation 24-8, subjects tended to consider the displayed outcome as correct when it had about 10 items, thus underestimating the result of the subtraction. They also confirmed that participants' responses displayed scalar variability. In the aforementioned study with 9-month old infants, McCrink and Wynn (2009) found that infants' expectations about the outcome of additions and subtractions also present operational momentum, suggesting that adults and infants recruit the same underlying mechanisms for non-symbolic calculation.

Animals are also able to perform non-symbolic calculations. In a violation-of-expectation experiment similar to those conducted by Wynn (1992a) with infants, Flombaum, Junge, and Hauser (2005) showed that rhesus monkeys that had not received any training formed expectations about the outcome of physical analogues of arithmetical operations such as 3+1, 2+2, and 4+4. Many other species, such as chickens (Rugani et al., 2009), pigeons (Brannon, Wusthoff, Gallistel, & Gibbon, 2001) and honeybees (Howard, Avarguès-Weber, Garcia, Greentree, & Dyer, 2019) have also been reported to be able to calculate sums and subtractions performed over collections of physical items.

Summing up, in this section we have seen that quantical abilities comprise the abilities to subitize, to estimate, and to non-symbolically calculate. Subitizing is fast and accurate, but limited to collections of up to three or four items. Estimation, in turn, does not have an upper limit, but is increasingly imprecise and is subject to Weber's law and adaptation, two psychophysical factors that affect the perception of many sensible magnitudes. Non-symbolic calculation, when involving small collections, can be as precise as subitizing, but with larger collections behaves as estimation.

There is a latent tension in quantical cognition. On the one hand, quantical cognition gives us subitizing, by means of which we can perceive small numerosities accurately. On the other hand, quantical cognition gives us estimation, which reveals that there are numerosities larger than the ones we perceive through subitizing, but of which it permits only a blurred perception. The challenge that pre-numerate humans who felt this tension had to face was how to extend the reliability of subitizing to treat the larger numerosities blurred by the psychophysical laws of perception.

3.2 The mechanisms underlying quantical abilities

Several cognitive models have been proposed to account for the non-symbolic numerical abilities displayed by humans and non-human animals. To date, the prevailing view is that humans and non-humans animals share two basic non-verbal mechanisms to process discrete quantities: a precise system for small quantities, called the Object Tracking System (OTS) or the Object File System (OFS) or parallel individuation, and an imprecise system for large quantities, called the Approximate Number System (ANS) (Knops, 2020).

The ANS is a domain-specific mechanism dedicated to the perception of numerosities, sometimes also called "the number sense" (Burr et al., 2017; Dehaene, 2011). There are many competing mathematical models of the ANS. The two most often cited are the linear model, proposed by Gallistel and Gelman (1992), and the logarithmic model, proposed by Dehaene and Changeux (1993). Both models make highly similar behavioral predictions (Dehaene, 2001) and possess equal ability to account for effects such as Weber's law, distance and size effects, and scalar variability. In both models, numerosities are represented as Gaussian curves on a metaphorical "mental number line," in which larger numerosities are represented by distributions that overlap increasingly with nearby numerosities. This increasing overlap is intended to account for the increasing noise that is observed in numerosity discrimination as numerosities grow. For example, in both models it is more difficult to distinguish nine from eight than four from three because the curves representing nine and eight overlap more than the curves representing four and three.

The two models differ, however, in the way they produce the increasing overlap of nearby numerosities. In the linear model, the number line is represented in the conventional fashion, as a series of equally spaced numbers, but the curves associated with each number become more and more stretched as numbers increase. In the logarithmic model, by contrast, the curves associated with each number have the same shape, but the number line is represented on a logarithmic scale, so that the space between numbers decreases as numbers increase, yielding a "compressed" number line. Figure 3.6 graphically compares both models.

As is evident from the graphs in Figure 3.6, in both models the curves of activation associated with small collections (less than four items) are easier to distinguish, allowing for the more precise enumeration of collections with one, two, or three items. This means that the

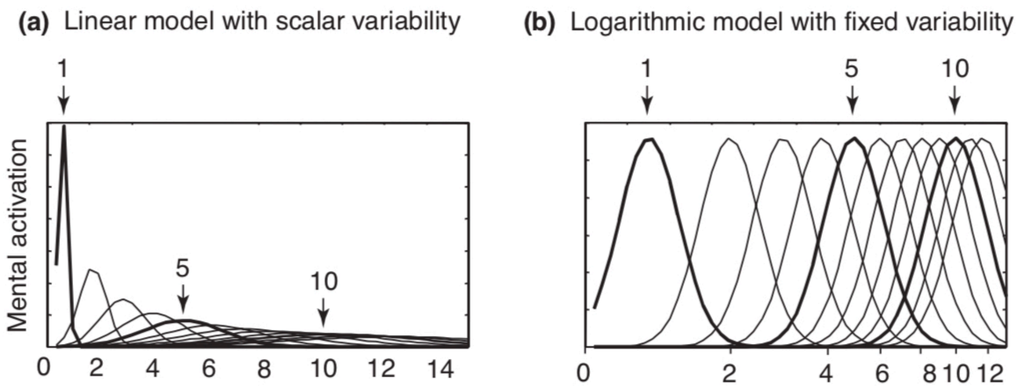

Figure 3.6: Two models of the ANS. In both models, the Gaussian curve associated with each number represents the mental activation produced by perceiving a collection with that number of objects. As numbers grow, the resulting mental activation becomes increasingly more similar, giving rise to an increasingly vague perception of numerosity. (Figure reproduced from Feigenson et al. (2004, p. 309).)

ANS could, at least in principle, account for both subitizing and estimation. However, as we have seen, behavioral evidence suggests that subitizing is produced by a different mechanism, since the psychophysical laws that govern estimation seem to play no role in subitizing. This was confirmed by Revkin, Piazza, Izard, Cohen, and Dehaene (2008) in an experiment designed specially to probe whether the ANS could account for the perception of numerosities within the subitizing range. They hypothesized that, if the ANS accounted for both small and large numerosities, then distinguishing 30 from 20, and 20 from 10, should be as easy as distinguishing three from two, and two from one, since in the ANS any two discrete quantities with the same ratio are equally easily distinguishable. But their results showed that it is much easier to distinguish between collections of one, two, and three items than between collections of 10, 20, and 30 items. There is virtually no error in numerosity discrimination within the subitizing range, whereas discrimination between collections of 10, 20, and 30 items is much more error-prone. Besides behavioral data, brain imaging studies have also detected different brain processes for small and large numerosities (see next section). These results have lent increasing support to the hypothesis that subitizing is not implemented by the ANS.

The accurate perception of small numerosities is often attributed to the Object File System (Trick & Pylyshyn, 1994). In contrast to the ANS, the OFS is not dedicated exclusively to the perception of numerosities. It is a domain-general mechanism for tracking multiple objects in space and time (also known as MOT: Multiple Object Tracking system) (Chesney & Haladjian, 2011). The OFS represents collections of objects by creating a working memory model in which each object is represented by a unique mental symbol. The OFS is believed to have a limited storage capacity, comprising only three or four "slots" for objects—and this is why subitizing is limited to three or four items.

Trick and Pylyshyn (1994) propose an explanation of how visual subitizing might take place in the OFS. They call each slot in the OFS a FINST (FINgers of INSTantiation). Each FINST acts as a pointer variable that points to an object in the visual display. Subitizing occurs in two steps. First, each object in the visual display is matched to a FINST, until all objects are matched or no more free FINSTs are available. This step is thought to be common to all non-verbal exact enumeration observed in infants, animals, and adults within the subitizing range. In numerate humans' verbal subitizing—when the subject utters a number word corresponding to the perceived numerosity—there is a second step, in which each occupied FINST is matched with a mental representation of a number word, in the usual order. In Trick and Pylyshyn's model, this second step accounts for the slight increase in reaction time observed from one to four in Figure 3.1.

A consequence of there being two systems responsible for numerosity perception is that numerosities with one to four items may be represented twice in the brain. Although the ANS alone cannot account for subitizing due to the aforementioned reasons, some argue that, at least in some situations, the ANS may take over the responsibility for representing small numerosities, or both systems may cooperate. Dehaene (2011, p. 258) maintains that the fact that monkeys trained to order collections of one to four items immediately generalize their methods to larger collections of up to nine items, as demonstrated by Brannon and Terrace (2000), counts as evidence for the double representation of small numerosities. Izard, Dehaene-Lambertz, and Dehaene (2008) showed in a neuroimaging study that both small (2-3) and large (4-12) numerosities can recruit the ANS in some situations. Hyde (2011) suggests that constraints on attentional and working memory resources might determine whether the OFS or the ANS is recruited for the enumeration of small numerosities. He proposes that when small collections of objects are presented in a way that precludes the OFS from encoding each object individually—e.g., too close together and under high attentional load—the ANS comes into play. Chesney and Haladjian (2011, p. 2478) propose that the visual indexes occupied in the OFS in a given moment could serve as an input for the ANS.

3.3 The neuronal bases of quantical cognition

There is limited knowledge about the neuronal mechanisms that implement the OFS and the ANS. However, there is plenty of evidence about where these systems are believed to be implemented in the brain. The parietal cortex, in particular the intraparietal sulcus (IPS), and regions in the prefrontal cortex have been identified as the most important areas for numerical processing in the human brain, both within and above the subitizing range (Castaldi, Vignaud, & Eger, 2020; Nieder, 2016) (see Figure 3.7). Various other regions of the brain, such as the inferior temporal lobe, inferior frontal regions, the medial temporal lobe, and the early visual cortex have also been related to the processing of quantical and numerical information (Wilkey & Ansari, 2020, p. 83).

Studies with infants and children have shown that most of the brain's quantical network is already in place from early infancy. Cantlon, Brannon, Carter, and Pelphrey (2006) found that the IPS responds to numerosity similarly in 4-year-old children as in adults, showing that the most important locus of numerical cognition in adults takes form prior to sophisticated symbolic numerical experience. Hyde, Boas, Blair, and Carey (2010) observed that the right inferior parietal occipital region in the brains of 6-month-old infants is already responsive to numerosity, showing that the specialization of this region for numerosity occurs before the acquisition of language.

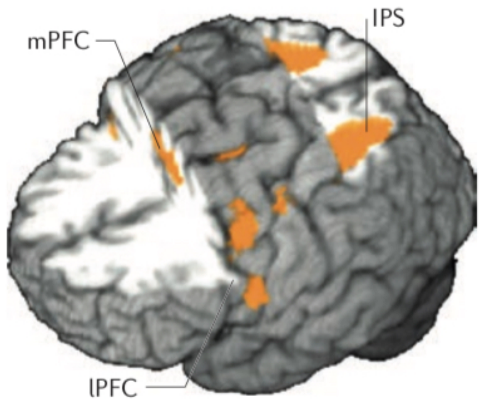

Figure 3.7: Number network in the human brain. The highlighted areas are regularly activated when subjects are performing numerical tasks during brain imaging studies. IPS: intraparietal sulcus; lPFC: lateral pre-frontal cortex; mPFC: medial pre-frontal cortex. (Figure reproduced from Nieder (2016, p. 368).)

In line with the behavioral evidence we have seen above, there is growing neuroimaging evidence that the processing of small and large numerosities is also functionally dissociated in the brain. Hyde and Spelke (2009) showed that small collections (1-3 items) evoke an early response in the posterior parietal area, whereas large collections (8-24 items) evoke a later response in this same area. Hyde and Spelke (2012) detected the engagement of anatomically distinct regions of the parietal cortex in processing small and large numerosities. Cutini, Scatturin, Moro, and Zorzi (2014) also observed a different pattern of activation in the parietal cortex for numerosities within and above the subitizing range, but only in terms of amplitude and temporal profile. Bloechle et al. (2018) showed that increase in numerosity within the subitizing range has no impact on the amplitude of the IPS response, whereas increase in numerosity above the subitizing range also increases the neural response in the IPS.

Regarding the neural implementation of the ANS "mental number line," Hyde and Spelke (2009) observed that neural responses for large numerosities accord with Weber's law. Lyons, Ansari, and Beilock (2015) showed that the curves of activation of neurons in IPS areas increasingly overlap as numerosity grows, confirming a prediction of ANS models. Perhaps the most impressive findings on the neural implementation of the ANS come from studies of the monkey brain. Recordings of the activity of single neurons in monkeys revealed that these neurons are attuned to specific numerosities (Nieder, 2016). The so-called "number neurons" respond most strongly to a preferred numerosity and also respond to a lesser extent to adjacent numerosities, in accordance with the predictions of ANS models. For example, a number neuron that responds with greatest intensity to an array of three dots, also responds with less intensity to two dots and four dots. Kutter, Bostroem, Elger, Mormann, and Nieder (2018) identified "number neurons" in the human brain. Harvey, Klein, Petridou, and Dumoulin (2013) identified populations of neurons in the right parietal cortex tuned to specific visual numerosities. These neurons are topographically organized, with populations tuned to smaller numerosities occupying a larger region than those tuned to larger numerosities.

In sum, investigations into the neuronal bases of quantical cognition have confirmed that there are two systems in the brain responsible for the implementation of quantical skills, and that these two systems are already in place before mathematical learning takes place. Moreover, similar structures have been found in non-human animals.

3.4 The evolutionary roots of quantical cognition

The fact that specialized brain structures associated with quantical cognition are in place at birth, along with the fact that quantical abilities are observed in many species—from honeybees to human beings—suggest that quantical cognition has evolutionary origins (Brannon & Merritt, 2011). Differences in the architecture of the brains of the species studied so far, on the other hand, suggest that quantical competence may not be due to a common ancestor, but rather to convergent evolution (Nieder, 2018). "The parallel co-evolution of numerosity sensitivity in these species underlines the idea that numerosity represents an important natural category to adapt behavior to environmental factors" (Knops, 2020, p. 49).

Many studies have shown that being able to perceive numerosity enhances animals' ability to survive, reproduce and compete for resources. Just a few examples: the ability to perceive numerosity helps mosquitofish to join the larger shoal and maximize the benefits of group protection against predators (Dadda, Piffer, Agrillo, & Bisazza, 2009); American coots have been shown to enumerate their own eggs in order to separate them from parasitic eggs laid by other birds in their nests (Lyon, 2003); the New Zealand robin's ability to perceive numerosity enables this food-hoarding bird to prioritize cache sites where more pieces of food are stored (Armstrong, Garland, & Burns, 2012; Hunt et al., 2008). Studies such as these support a general consensus in the literature on numerical cognition that the ability to perceive numerosity is adaptive (Nieder, 2018).

When it comes to primates, besides the obvious survival advantages of quantical abilities, Cantlon (2018) proposes that primates' abilities result from the joint satisfaction of two evolutionary constraints. First, she claims, "number" is the optimal solution to the problem of integrating quantity information across modalities. Other quantitative dimensions, such as length, surface area, weight, rate, and loudness, are modality-specific and, therefore, cannot be readily used to compare collections of objects across different modalities. For example, it is not possible to assess length by hearing, or loudness by seeing. Numerosity, by contrast, can be perceived and compared over different modalities. Second, primates' perceptual systems' tendency to segment stimuli in discrete objects might have made numerosity a salient trait of the environment for them. In her words, numerosity perception is "an object-based representation ideally suited to the object-based nature of primates' visual processing" (Cantlon, 2018, p. 66).

It is easy to imagine that the same constraints may have contributed to the emergence of numerosity perception in other species. Corvids, for example, have highly developed capacities to represent objects (Hoffmann, Rüttler, & Nieder, 2011) which could make numerosity a dimension as salient for them as it is for primates. Although dissimilarities between the corvid and the primate brains suggest that numerical competence in corvids and primates is not due to a common ancestor, behavioral and neuroimaging data show not only that both species share very similar abilities, but also that both have developed a similar strategy to compute numerosity, captured by ANS models (Nieder, 2018). This "suggests that this way of coding numerical information has evolved based on convergent evolution because it exhibits a superior solution to a common computational problem" (Nieder, 2018, p. 9).

Paying attention to the evolutionary roots of quantical cognition may help philosophers avoid misinterpreting the philosophical significance of the fact that these abilities are innate. Based on Wynn's studies of non-symbolic calculation in infants (mentioned above in section 3.1.2), Gaeta (2017) claims that findings from numerical cognition vindicate the rationalist thesis according to which mathematical knowledge is a priori. Gaeta writes:

> Benacerraf (1973) has raised a problem about mathematical knowledge: how can the abstract entities that are the subject matter of mathematical knowledge can [sic] be known to human beings of flesh and blood if it is impossible for those entities to exert any causal relation on them? The question presupposes a causal theory of knowledge, like Locke's doctrine about the effect on the subject by external objects. But the scientific results outlined in the preceding section show that this is not the only way, and not the most correct one, to formulate the question. Apart from experience and physical causality there may be other ways of forming beliefs that, in principle, constitute knowledge: they work perfectly in our dealings with the world and are universally recognized. Nevertheless, its origins are not located in the external material objects that stimulate our senses but in our genes. Perhaps the old rationalist philosophers would make a face of resignation in front of a situation that is still paradoxical. After so many centuries, the new generations of their eternal rivals, the empiricists, seem to have discovered by just taking advantage of the own methods of the empirical sciences that innate knowledge, that is, a priori knowledge, could be more than a mere illusion (Gaeta, 2017, p. 221).

Although Gaeta is right in claiming that quantical cognition is codified in the genes and hence innate, this does not mean that quantical cognition gives us any kind of a priori knowledge. 'Innate' means present at birth. 'A priori' means independent of experience. Quantical cognition is innate but not a priori because it was shaped by environmental pressures acting upon our ancestors. Traits evolutionarily selected because of their adaptive advantages are not independent of experience. Quantical cognition evolved in the way it did because the experiences of our ancestors were such that they favored the survival and reproduction of individuals who had those abilities. Quantical cognition is the product of experiences accumulated through generations and it is, therefore, a posteriori.

Philosophical misinterpretations of findings from numerical cognition are not uncommon. Sometimes, however, philosophers' mistakes might be motivated by the loose way scientists use philosophically-relevant terms such as number, cardinality, and numerosity. In the next section, I carefully disentangle these three concepts so that we have a clearer idea of what things and concepts are involved in quantical cognition.

3.5 The ontology of quantical cognition

In the previous sections, I have used expressions such as "number sense," "approximate number system," "mental number line," and "number neurons." These expressions, whose use is widespread in the literature on quantical cognition, give a sample of how pervasive the idea is that the abilities described above are genuine *numerical* abilities. These abilities are believed to be numerical in two senses. On the one hand, some cognitive scientists speak as if subitizing and estimation enable us (and infants and honeybees) to *see* numbers; numbers would be a perceptible property of the environment. Some examples:

> Newborn infants perceive abstract *numbers* (Izard, Sann, Spelke, & Streri, 2009, title, emphasis added).

> [A] great deal of evidence suggests that humans perceive *number* spontaneously, with dedicated mechanisms (Burr et al., 2017, p. 9, emphasis added).

> Ample evidence from single-cell recordings, psychophysics, and brain imaging studies suggests that humans and animals perceive *numbers* innately (Nieder, 2019, p. 147-148, emphasis added).

> We and animals can represent *cardinal numbers* because they are a significant feature of the world, and assessing number increases fitness and survival (Nieder, 2019, p. 28, emphasis added).

> *Number* As a Primary Perceptual Attribute: A Review (Anobile, Cicchini, & Burr, 2016, title, emphasis added).

On the other hand, some cognitive scientists speak as if the cerebral implementation of subitizing and estimation involves numbers or representations of numbers in the brain. In this view, numbers (or representations thereof) would be innate brain resources.

> Behavioural and electrophysiological data now convincingly establish the existence of *numbers* in the brain—in animals from insects to humans (Gallistel, 2017, p. 1, emphasis added).

> Although humans may be the only species with a linguistically mediated code for numbers, we share an approximate, non-verbal *representation of number* with many animal species, as many papers in this special issue make amply clear (Burr et al., 2017, p. 1, emphasis added).

> Representation of *Number* in Animals and Humans: A Neural Model (Verguts & Fias, 2004, title, emphasis added).

Many times these two ways of thinking of numbers as involved in quantical cognition appear together: the existence of numbers in the environment would give rise to representations of numbers in the brain. This is how Nieder conceives of "number neurons," the "neurobiological foundations" of the ability to perceive numbers, which are (or so he claims) "a property of real objects and events" (Nieder, 2019, p. 5;148).

Philosophers of mathematics with a platonist leaning would be appalled in face of these declarations. Philosophically, the distinction between having numbers in the brain and having number *representations* in the brain is highly relevant. If we have only number representations in the brain, then numbers themselves must be somewhere else; whereas if we have numbers themselves in the brain, this means that numbers *are* mental entities. This latter claim is untenable, as Frege (1960) has already shown. The claim that there are representations of numbers in the brain is philosophically less problematic, but these cannot be representations of a property of the environment, because number is not a property of aggregates of matter, as Frege (1960) also argued (I present Frege's arguments below).

There is another term commonly used in the literature on numerical cognition that can help make sense, from a philosophical point of view, of what is really involved in quantical cognition. The term *numerosity* was originally introduced to allow cognitive scientists to "[make] fine distinctions to properly evaluate and measure stimuli—especially when studying rats, pigeons, and infants—without having to necessarily assume the presence of conceptual understanding such as that involved in the notion of number" (Núñez, 2017, p. 410). Numerosity is usually defined as a synonym of cardinality:

> Cardinality (also known as numerosity) corresponds to the empirical property of quantity, and is the number of countable elements in a given group (for example, five runners) (Nieder, 2016, p. 366).

> Numerosity, the cardinality of a set, is a property that applies to any set of individual objects (Piazza & Izard, 2009, p. 261).

> ... the number of things in a set—the numerosity of a set. (The term 'numerosity' is used here as the cognitive counterpart to the term 'cardinality' used by mathematicians and logicians) (Butterworth, 2005, p. 3).

The definition of numerosity as cardinality is inadequate, as I will argue in section 3.5.3, but at least the distinction between numerosity and number allows for a more careful way of speaking, according to which infants, pigeons, and honeybees perceive *numerosities*, have innate representations of *numerosities*, and thus are not necessarily held to be competent with numbers. However, as Núñez notices, the term "numerosity" (and other terms originally introduced to allow for finer conceptual distinctions) is not consistently employed:

> the field of numerical cognition has been notorious for not employing precise terminology when dealing with the concept of number. Already three decades ago scholars investigating 'numerical competence' in nonhuman animals and children spoke of the 'terminological chaos' ([44] p. 562), the lack of 'clarification of terms' ([73] p. 601), and the unnecessary suffering 'from the misapplication of terms' ([74] p. 580) that existed in the field. The situation is no better today because relatively precise definitions of various number-related terms—some of which were carefully coined by the psychophysicists of the mid-20th century [75,76]—are routinely blurred (Box 1). Sometimes 'number' is used to mean 'numeral' (e.g., [77]), or sometimes 'numerousness' (e.g., [78]), despite warnings that 'numerousness discrimination ... represents a simple perceptual ability that bears no obvious relation to number' ([79] p. 1222). More importantly, 'number' is often loosely used in place of 'numerosity'. Articles in developmental (e.g., [80] p. B15) and comparative (e.g., [45] p. 86) psychology while properly discussing 'numerosity' when describing stimuli, leap to 'number' in conclusions. Similar loose inter-changeability can be found in neuroscience publications (e.g., [81] p. 177) (Núñez, 2017, p. 417).

This loose and misleading use of terminology prevents us from reading from the scientific texts an ontology of quantical cognition. Scientists themselves seem to be confused not only about the use of the relevant technical terms of their field, but also about the entities that are involved in quantical cognition. This "chaos" is a unique opportunity for philosophical conceptual analysis to contribute to scientific understanding by improving conceptual clarity, and a sine qua non for the account of the nature of numbers I am pursuing here.

In the remainder of this section, I seek the conceptual clarity necessary to understand quantical cognition and its relation with numbers and arithmetic in general. In doing so, I will not question the empirical findings of the field, nor discuss the suitability of the proposed models. These are empirical matters, not subject to objections from conceptual analysis. I will instead question the scientific discourse about these findings and models. The following are the questions I want answered:

1. What do we perceive through subitizing and estimation? Do we perceive numbers?
2. What is involved in the internal mechanisms that implement quantical skills? Do we have numbers in the brain?

In the following two subsections I address each question in turn.

3.5.1 What we perceive through quantical skills

In scientific texts, as exemplified above, it is said that subitizing and estimation enable us to perceive numbers, numerosities, or the cardinality of collections. Notwithstanding the fact that scientists carelessly use these terms interchangeably, they have different meanings. 'Numerosity' is a technical term from numerical cognition. The terms 'number' and 'cardinality,' in turn, belong to mathematics, where they refer to different concepts. In mathematics, the cardinality of a set refers to its "size," and it is a trivial fact that the cardinal size of a set can be determined without even mentioning numbers. This is easily illustrated as follows. Imagine we want to know whether the cardinality of the set of people in a room is equal to the cardinality of the set of chairs in that room. We do not need to count people and chairs. We can just ask people to sit down, each person in a single chair. If no person remains standing and no chair remains empty, we conclude immediately that both sets have the same size. If someone remains standing, then the set of people is larger than the set of chairs; inversely, if any chair remains empty, then the set of chairs is larger than the set of people. No number is involved in this procedure. It can be carried out recruiting only the notion of one-to-one correspondence or equinumerosity which, despite its name, is defined without invoking any numerical concept (Enderton, 1977, p. 129).

True enough, numbers are used to evaluate and express cardinality. But in principle we can use any set as a "yardstick" to do the same by establishing one-to-one mappings between its elements and the elements of the set we want to evaluate the cardinality of. For example, I can use the set of pens on my desk—let us call it P—for this purpose. Because there is an injective and non-surjective mapping between P and the set of sections in this chapter, I can say that the cardinality of the set of sections in this chapter is greater than P's. The existence of a bijective mapping between P and the set of pens on my table allows me to say that the cardinality of the set of pens on my table is equal to P's. P is not a number, but I just used it to express cardinalities in the same way I could have used the number two. One obvious advantage of using numbers is that almost everyone knows which cardinality corresponds to two, whereas only I know the cardinality of P. Moreover, P can express only one cardinality with precision, whereas we have numbers to express every cardinality with precision. These shortcomings, however, do not prevent P from working as a cardinality ruler.

Mathematically, the concept of cardinality is independent of the concept of number. This is not to imply that these concepts are completely unrelated, of course. On the contrary, the cardinal value expressed by each number has served as inspiration for strategies to define them. Thus, in von Neumann's approach, each number is defined as a set whose cardinality corresponds to the cardinal value of the number. For example, two is the set $\{\emptyset, \{\emptyset\}\}$, whose cardinality is two. In Frege's (failed) approach, each number is defined (roughly) as the class of all sets whose cardinality corresponds to the cardinal value of the number. For example, two is defined as the class of all sets that, for some distinct x and y, have as members x and y and nothing else (Enderton, 1977, p. 125). But notice that in both von Neumann's and Frege's definitions the concept of cardinality does not appear in the definiens; the idea that to each number there corresponds a cardinal value only intuitively motivates the definitions. It is not indispensable for a definition of number in any sense. There are other approaches, such as Zermelo's, in which the cardinality of the set that defines a number, as well as the cardinality of the elements of this set, simply does not matter. For example, in Zermelo's approach two is defined as $\{\{\emptyset\}\}$, three is defined as $\{\{\{\emptyset\}\}\}$, although both sets have cardinality one. In sum, although number and cardinality are related, they are independent from each other in terms of their mathematical definitions.

With this distinction in place, it is easy to see that in quantical cognition experiments, what is tested is participants' ability to perceive cardinalities (the size of a collection) or numerosities, not numbers. Subjects tested in these studies are shown to perceive *the number of things* they are presented with, not number *simpliciter*. The expression 'the number of things,' despite containing the word 'number,' is not a synonym of number, but a synonym of cardinality. If a participant is shown to be able to distinguish ♦♦ from ♦♦♦, then we may conclude that she can perceive a property of these collections, namely, their cardinality (provided that the experiment controlled for other variables, such as length). We do not have any reason to believe that she saw numbers.

The confusion here is understandable. Because we use numbers to measure cardinalities, we usually refer to cardinalities indirectly, via their numerical measurements. For example, in the sentence 'the number of planets is eight,' the expression 'the number of planets' is used in place of 'the cardinality of the set of planets.' This is a clear case of metonymy, the figure of speech in which a word is used in place of another associated with it. A similar case of metonymy occurs when we say that "the square footage of the house is 1,200 sq ft." In this case, the expression 'the square footage of the house' is used in place of 'the area of the house,' which is the property that is being measured in square feet. Thus, when an infant is said to be able to perceive "the number of dots on the screen," what this really means is that the infant is able to perceive the cardinality (or, more precisely, the numerosity, as we will see below) of the set of dots on the screen, which is the property that is being tested in the experiment. It may be that the infant is using numbers to evaluate cardinality, but certainly she is not seeing numbers, as we do not see square feet when we attend to the area of a house.

Some philosophers have contested the distinction between cardinality and number, or between numerosity and number. Against the latter, Jones (2018, p. 152) argues that

> one might object that the ANS is not a system for numerical perception, since most "cognitive scientists distinguish between *numerosities*, the concrete, discrete magnitudes that animals represent, and *numbers*, the abstract entities that are studied by

> mathematicians and philosophers of mathematics" (De Cruz, 2016, 3). In other words, we perceive numerosities, not numbers. However, this distinction is not tenable in the current context since assuming that there is a distinction between *concrete* numerosities and *abstract* numbers involves assuming the impossibility of numerical perception from the outset.

Certainly, we should not assume from the outset, as I have argued in Chapter 2, that numbers are abstract. But the hypothetical non-spatiotemporality of numbers is not the problem here. The problem is that, even if numbers are perceivable in any sense, they do not show up in the experimental setups used in quantical cognition studies. Consider, for example, the stimuli presented in Figure 3.8, used in Izard's et al. (2009) study with infants. The stimuli consist of geometrical figures with small faces depicted in them. If infants see these as making up a collection, then this collection has cardinality as one of its properties, and infants can grasp it. That is what there is to be seen in such a situation. Perhaps infants are using some numerical understanding in doing so, but they are definitely not perceiving numbers.

The only way to maintain that participants are seeing numbers themselves in such situations would be by conflating the concepts of cardinality and number. However, this not only flies in the face of mathematics but would also backfire, since the conflated property number/cardinality would have to be non-spatiotemporal and, hence, unobservable. This is made clear in Giaquinto's (2001; 2017) platonist account of numbers. In Giaquinto's view, cardinal numbers are set sizes (i.e., cardinalities), but he acknowledges that

> the set-size view of cardinal numbers runs into the cognitive access problem ... If numbers were set sizes, they would lack space-time location; they could not undergo any change; they could neither emit nor reflect signals; they could leave no traces; they could not affect the behaviour of other things. So they could have no causal effect on us, even remotely. So we could have no cognitive access to them (Giaquinto, 2017, p. 2).

In the same paper, Giaquinto argues that numerical cognition allows us to circumvent Benacerraf's Problem. An evaluation of Giaquinto's arguments is beyond the scope of this section, but it is relevant for us to see why the set-size view of cardinal numbers implies that numbers are non-spatiotemporal. The reasoning goes as follows.

Consider again the set of pens on my desk, P. The elements of P are localized in space and time. But where is P, the set, and its properties? If the set and its properties are localized in the same place and time as its elements, then its number is localized on my desk. Since P

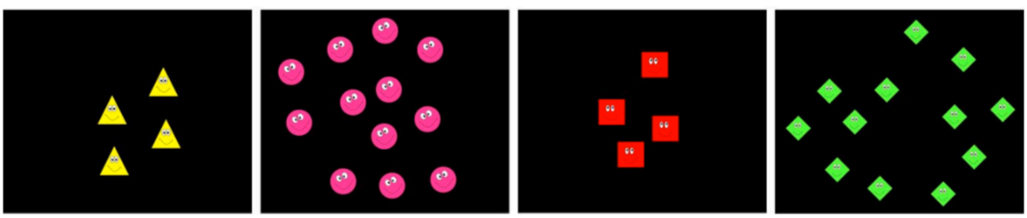

Figure 3.8: Stimuli used in experiment testing numerosity perception in newborns. (Figure reproduced from Izard et al. (2009, p. 10383).)

has two elements, I have a number two on my desk. But I also have a number two on my face, given that I have two eyes, and yet another number two in my ears. This means that there are as many numbers two as collections with two elements in the universe. This conclusion is plainly unacceptable. It runs against mathematics, where there is only one of each number.[6] Moreover, numbers do not have geographical and temporal properties. Therefore, we must reject the assumption that sets and its properties are localized in space and time. As a result, we have to accept that sets and their properties are non-spatiotemporal. This leads to the well-known view according to which numbers are abstract universals, and each particular set or collection of physical things instantiates the universal corresponding to its number. In this case, *P* would be an *instance* of two, but two itself would be a non-spatiotemporal universal. In this case, at most, we could perceive *instances of numbers*, but not numbers directly.

However, we do not have any reason to assume this conflation and its consequences. It leads us back to platonist puzzles we are trying to avoid. Furthermore, from the outset, the conflation of number and cardinality goes against the mathematical distinction between the two concepts, thus it is not surprising that it gives rise to philosophical troubles. This is sufficient reason to keep cardinality and numbers as distinct concepts.

With this distinction in place, it becomes clear that numbers are not perceived through quantical skills, even if numbers were perceivable in any sense. It is still too premature to make this affirmation here, but my point is that numbers, as symbolic cognitive tools, are not like objects we can look at. They are involved in a method—counting—we use to assess cardinalities precisely, but counting is not the only method of doing so. We can also assess cardinalities by one-to-one correspondence (tallying) and by subitizing. We have at least three methods of assessing cardinalities at our disposal, each one with its own advantages and limitations.

So far, it seems that cardinalities are what we perceive through our quantical skills. This is not so, however, since there are good reasons to distinguish between cardinality and numerosity. One reason is that the latter is a clear-cut attribute, whereas the former is not. The cardinality of a set with 53 elements is clearly distinguishable from the cardinality of a set with 54 elements. However, our quantical skills do not enable us to tell the difference between the cardinalities of these sets. For this reason, above the subitizing range it is not adequate to say that we can perceive cardinality. At best, we can perceive *approximate* cardinalities, and approximate cardinality is numerosity. This point is suggested by Dehaene (2011, p. 23).

> The difference with our verbal counting is so enormous that we should perhaps not talk about "number" in animals at all, because by number we often imply a discrete symbol. This is why scientists, when they describe perception of numerical quantities, speak of "numerosity" or "numerousness" rather than number. The accumulator [the ANS] enables animals to estimate how numerous some events are, but does not allow

[6] As Giaquinto (2017, p. 2) observes, "this can be seen from the use [in mathematics] of number-counting functions, the definitions of which make no sense unless it is assumed that there is just one number per numeral." Giaquinto mentions as an example Euler's ϕ function, which counts the positive integers up to a given integer n that are co-prime to n (i.e., $\phi(1) = 1$ and $\phi(n) =$ the number of positive integers less than and co-prime to n, for $n > 1$).

> them to compute their exact number. The animal mind can retain only fuzzy numbers.

In this passage, Dehaene uses ‘number’ in the metonymic sense I discussed above. His point seems to be that we should not talk about perception of cardinality in animals because cardinality implies an exact discrete quantity, but animals can perceive only fuzzy numerosities. Numerosity is vague. Cardinality is exact. This is not to deny that numerosity and cardinality are intimately related. Up to the subitizing limit, perception of numerosity and evaluation of cardinality coincide. Above this point, they start diverging, and the divergence is governed by the psychophysical laws of perception which affect numerosity perception. Thus, we can provisionally say that numerosity is “perceived cardinality,” i.e., a subjective, relative, approximate judgment of cardinality.

The above considerations recommend restoring the unequivocal use of the term ‘numerosity’ to refer to what we can perceive through our quantical skills. In section 3.5.3, I give a more adequate definition of numerosities, but first we have to consider the role of numbers in the implementation of numerosity perception in the brain.

3.5.2 Numbers in the internal mechanisms of quantical cognition?

In the previous subsection I concluded that we do not see numbers through quantical skills, but I left open the question of whether or not our quantical skills use numbers to assess numerosities. As we saw above, scientists’ explanations of the models proposed to account for the mechanisms underlying quantical skills usually mention numbers or number representations in the brain. In this subsection, we will see that, in spite of scientists’ discourse, the very models they propose make clear that neither numbers nor number representations are involved in the implementation of quantical cognition.

First of all, it is important to notice that, conceptually, the perception of numerosity does not require numerical competence. Insofar as numerosity is perceived cardinality, it can be evaluated by one-to-one correspondence. And this is exactly how the OFS tracks numerosities, according to Trick and Pylyshyn’s (1994) model mentioned in section 3.2. The OFS represents numerosities by establishing a one-to-one mapping between the perceived objects and its memory slots, following a procedure similar to the one I exemplified above with the set of pens on my table. Cognitive scientists acknowledge that such mappings cannot count as number representations.

> The object tracking system … is thought to represent numerical information only in an implicit way: in this system, there is no summary representation of ‘two’; instead, infants form a mental model of two objects by recruiting two attentional indexes or ‘object files’ (Izard et al., 2008, p. 281).

Based on this feature of the OFS, Simon (1997) contends that subitizing in infants and non-human animals is best conceived of as *non-numerical*, because the discrimination of numerosities within the subitizing range demands nothing more than establishing one-to-one mappings and making same/different discriminations. To illustrate Simon’s point, let these signs represent the OFS’s memory slots of a subject: ◊◊◊◊. When the subject is presented with a collection of two objects, two slots are occupied and this information is stored in its memory: ♦♦◊◊. After that, when presented with a collection of three objects, three slots are occupied: ◆◆◆◇. To tell the difference between their numerosities, the only thing the subject has to do is to compare whether ◆◆◇◇ and ◆◆◆◇ are the same or different. The subject does not need to have any numerical understanding to do so. Thus, strictly speaking, subitizing is non-numerical. Subitizing becomes numerical only when numerate humans map OFS's working memory representations into number words and express the evaluated cardinality by uttering a numeral. But for innumerate humans and non-human animals, which do not take this last step, subitizing is non-numerical.

As for the ANS, Dehaene (2003, p. 145) literally speaks of a "logarithmic mental number line" in which "mental representations of number" are to be found: "this approximation mode is the natural way that *number* is encoded in a brain without language" (Dehaene, 2003, p. 147, emphasis added). Gallistel and Gelman (2000, p. 59) suggest that "the non-verbal representatives of number are mental magnitudes (real numbers) with scalar variability." The claim that there are numbers or representations of numbers in the internal mechanisms of the ANS, however, does not seem plausible. A basic property of numbers is that, given any number and its successor, the difference between them is exactly one. This feature is not preserved in the logarithmic mental number line proposed by Dehaene. Furthermore, if numbers were used to evaluate numerosities (by counting), outcomes would be fully accurate, except for occasional mistakes. However, ANS estimates are systematically inaccurate and subject to disturbances that do not affect numerical counting, such as the effects of Weber's law and sensory adaptation. Because numbers are exact by definition, Núñez (2017) regards the phrase 'Approximate Number System' as an oxymoron; he proposes instead naming this cognitive mechanism as Large Quantity Discrimination (LQD).

The fact that the ANS delivers fuzzy estimates suggests that it uses a continuous mechanism to assess numerosity, which does not square with the discreteness of natural numbers. That is why Gallistel and Gelman say, in the passage quoted a few lines above, that the mental magnitudes believed to represent numbers in the brain are "real numbers." The point, however, is that the continuous mechanisms underlying the ANS do not need to be numerical in any sense. The very explanation Gallistel and Gelman give of these mechanisms shows that it is non-numerical. According to them, numerosities are represented in the brain by means of analog representations similar to histograms.

> A histogram is a familiar example of the use of magnitudes to represent numerosities: the higher the column in a histogram, the greater the numerosity of the set represented by that column ... The column that represents the combined numerosity of sets 1 and 2 is the column you get by placing the column for set 1 on top of the column for set 2. The column that represents the more numerous of two sets is the first column contacted by a horizontal line lowered from the top of the graph. If you form a rectangle whose height is that of column 1 and whose width is the height of column 2, then hold constant the area of the rectangle while adjusting its width to the standard column width, you get a column whose height represents the numerosity of the set formed by multiplying the numerosities represented by columns 1 and 2.
>
> The system just described—histogram arithmetic or the histogrammic calculator—is an analog system isomorphic to arithmetic. Its symbols are magnitudes, the heights of the columns. Its operations are processes involving magnitudes ... We want to suggest, on the basis of both animal and the human data, that the preverbal processes that underlie both the animal and the human capacity to represent numerosities and rea-

> son arithmetically are analogous to histogram arithmetic. These preverbal processes generate analog mental variables (ultimately, of course, neurophysiological variables) that function as the mental/neural representatives of numerosity (Gallistel & Gelman, 1992, p. 46).

In the above passage, Gallistel and Gelman explain an ingenious non-numeric method of calculation. Basically, instead of using cardinal numbers to represent cardinalities or numerosities, one can use columns on a histogram; instead of performing arithmetical operations such as additions and multiplications, one can manipulate the columns. The mechanism that produces a column to represent a given numerosity is also non-numerical. This mechanism, an accumulator, was originally proposed by Meck and Church (1983). Roughly, as Gallistel and Gelman (2000) explain it, the accumulator is like a beaker into which one cup of water is poured for each item in a given numerosity. The final level of the column of water in the beaker represents the size of the collection. Clearly, the accumulator implements a non-numerical process of one-to-one correspondence.[7]

Dehaene (2011, p. 17-19) also uses the accumulator mechanism to explain how the ANS works. In contrast to Gallistel and Gelman's accumulator, though, Dehaene's accumulator introduces "noise" in the representation of numerosities from the outset. To each item in the to-be-evaluated numerosity, a slightly variable amount of water is poured in the beaker. The larger the collection, the larger the accumulated noise, and therefore the one-to-one correspondence between "cups" in the beaker and items in the collections is progressively blurred. Thus, Dehaene's ANS can be seen as implementing a third way of approximately assessing numerosities, which does not rely on either numbers or one-to-one correspondence.

If these models are correct, then what cognitive scientists call "number sense" is, in fact, non-numerical. All its processes are explained, in the scientific models themselves, by mechanisms that do not involve any conceptual contribution of numbers. The irony is that the very same cognitive scientists who have advanced these models talk as if they involved numbers, or as if they enabled us to see numbers. In my opinion, this is due to the fact that we are so used to relying on numbers to assess cardinalities that we cannot help talking about numbers when speaking of evaluation of cardinalities (or numerosities). But once it is clear that there are non-numerical methods of assessing cardinalities, it becomes evident that the mechanisms underlying the OFS and the ANS may be non-numerical. Núñez proposes the neologism "quantical"—which I have been using since Chapter 2—to qualify these processes and the corresponding abilities they give rise to.

> The English language (like others) does not have an adjective to label phenomena that are quantity-related but lack the properties [of numbers]. One possibility would be to call them quantitative capacities [60], but this term—usually contrasted with 'qualitative'—relates to measurements and their numerical and mathematical treatment. I propose to refer to these biologically endowed capacities as quantical—in contrast to 'numerical' (Núñez, 2017, p. 419).

[7]In non-numerical one-to-one correspondence, a correspondence is established between two collections whose elements are not numbers. In the accumulator, these are "cups of water" and the elements of a collection of physical objects. Counting is also a form of one-to-one correspondence, but a numerical one, in which the elements of the given collection are paired with numbers.

In Núñez's terminology, preverbal or non-symbolic numerical cognition becomes *quantical* cognition, and non-symbolic numerical skills, such as those displayed by infants and non-human animals, become *quantical* skills.

Before concluding this section, we have to consider another way in which numbers could be seen as crucially involved in quantical cognition. It is a fact that all the scientific models of the ANS use numbers and other mathematical resources in their formulations. For example, in both models we considered in section 3.2, numbers label the activation curves associated with the perception of numerosities. Does this not show that numbers and other mathematical resources are indispensable for modeling quantical cognition? Does this not show that even quantical cognition, with its non-numerical methods of assessing numerosity, must presuppose the existence of numbers?

This question does not seem to concern scientists, but philosophers have pondered it. De Cruz (2016) makes a case for realism about numbers based exactly on this. She mentions the fact that "[s]cientific practice suggests a crucial explanatory role for numbers in research on numerical cognition" (De Cruz, 2016, p. 6), providing "a prima facie realist case for numbers, since neuroscientists and cognitive psychologists are interested in isolating numerical properties of the environment, and since they refer to numbers in their explanations" (De Cruz, 2016, p. 7). However, De Cruz recognizes that it is highly unlikely that numbers themselves take part in the implementation of the cognitive processes that support quantical abilities. She views numbers as acausal abstract entities, and thus they cannot take part in causal explanations of cognitive mechanisms. However, she argues, scientists have to assume the existence of these acausal entities in order to explain numerosity perception. In other words, numbers are indispensable for the scientific models of quantical cognition. De Cruz's realist case for numbers is an instance of the traditional Quine-Putnam Indispensability Argument.

Even if De Cruz might be right in claiming that numbers are indispensable for scientific models of quantical cognition, such indispensability is not capable of supporting a stronger case for number realism than other indispensable uses of numbers in science. Numbers are as indispensable for the explanation of quantical cognition as they are for the explanation of the movement of the stars. To the extent that the indispensability of numbers in quantical cognition models can also be reconciled with nominalistic or fictionalistic reinterpretations of mathematical language, De Cruz's case for realism is as inconclusive as the approaches we saw in Chapter 1. Now, if numbers are cognitive tools, as I am proposing, there is nothing special about the use of numbers in scientific models of quantical cognition. As cognitive tools, numbers can be used to model any kind of phenomena, from the movement of the stars to quantical cognition.

3.5.3 A closer look into numerosities

At the end of subsection 3.5.1, I provisionally defined numerosity as perceived cardinality. For reasons that I will make clear in this section, this definition is unsatisfactory. The problem is that this definition does not take into account a very basic, well-known fact about the concept of cardinality already pointed out by Frege: there is no unique characteristic manner in which an aggregate of matter can be separated into discrete parts so as to be counted.

> If I give someone a stone with the words: Find the weight of this, I have given him precisely the object he is to investigate. But if I place a pile of playing cards in his hands with the words: Find the Number of these, this does not tell him whether I wish to know the number of cards, or of complete packs of cards, or even say of honour cards at skat. To have given him the pile in his hands is not yet to have given him completely the object he is to investigate; I must add some further word—cards, or packs, or honours (Frege, 1960, §22, p. 28).

A deck of cards does not have number as a property of its own in the same way that it has weight as a property of its own. This has an important consequence for how we should interpret the results of experiments probing infants' ability to perceive numerosity. When an experimenter presents an infant with stimuli like the ones reproduced in Figure 3.8, there is no cardinality to be perceived in the stimuli themselves. The cardinality attributed to them will depend on what is to be counted as a unit. In the leftmost picture, for example, one can count four triangles, or eight eyes, or twelve eyes and mouths. If newborns attribute some particular cardinality to such a stimulus, this will depend on what they are counting as a unit. It may well be that infants will count each geometrical figure as a unit, as it seems to be the experimenter's intended criterion of individuation. However, when cognitive scientists say that numerosity—viewed as a synonym of *cardinality*—is a perceptible property of the environment, they are downplaying the role of cognitive agents in determining a criterion of individuation. A cardinality cannot be attributed to a stimulus unless an agent has already determined what is to be counted as the relevant unit. But then the cardinality so determined will not be a property of the stimulus, but a property of how the agent processed the stimulus.

In fact, cognitive scientists are aware of this. Here again we encounter a careless use of terminology, rather than a real conceptual mistake. For example, Wynn addresses this point, citing Frege explicitly, when she notices the necessity of a sortal to individuate and subsequently enumerate events or actions:

> actions are interesting to study because Frege's (1884/1960) point that number is not an inherent property of portions of the world applies every bit as much to actions as it does to physical matter. Just as a given physical aggregate may accurately be described as *52 cards*, *1 deck*, or *10^{28} molecules*, if I walk across a room it counts as 1 "crossing-the-room" action, but may count as 5 "taking-a-step" actions. Number of actions is only determinate relative to a sortal term that identifies a specific kind of action. There is no inherent, objective fact of the matter as to where, in the continuously evolving scene, one 'action' ends and the next 'action' begins. The individuation of discrete actions from this continuous scene is a cognitive imposition (Wynn, 2018, p. 109).

> [T]he enumeration mechanism may take as input conceptually specified entities—entities that the cognitive system conceptualizes as individuals (Wynn, 1996, p. 169).

When it comes to events, the need for a sortal is still more self-evident than in the case of static stimuli. But in this case as well, cognitive scientists recognize, albeit less explicitly, that some process of individuation is needed before numerosity can be assessed. For example, Cantlon (2018, p. 66) says that "an object is a *cognitive construct* recognized across modalities" (emphasis added), and Fornaciai, Cicchini, and Burr (2016, p. 66) write that "numerosity perception depends on segmentation of the elements in perceptual objects, following several rules such similar shape, orientation, common fate, and connectedness." In

Dehaene and Changeux's (1993) model of the ANS, as can be noticed in the schema depicted in Figure 3.9, between the input "retina," as the authors call it, and "numerosity detectors," there is a layer in which visual or auditory stimuli are segmented into objects. Notice that the authors represent the visual input as a continuous wave, suggesting that collections of discrete items do not come directly from sensory stimuli. It is the intermediate layer that segregates input stimuli into objects and groups them into collections. Only then can these collections, formed in the mind, have their numerosity assessed.

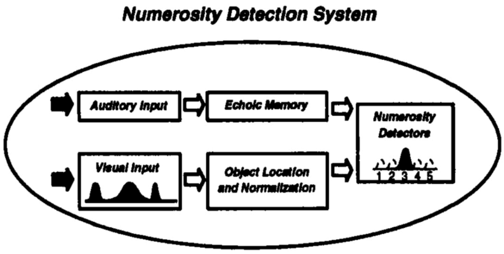

Figure 3.9: The intermediate layer between stimulus input and numerosity detection is where the individuation of objects takes place. (Figure reproduced from Dehaene and Changeux (1993, p. 394).)

As opposed to the freedom we have to choose a sortal when evaluating cardinality, when it comes to numerosity perception we have limited control of the processes of individuation and grouping of objects carried out by our perceptual system. The extent to which perception of numerosity is unresponsive to attempts to control it is illustrated by the effects of adaptation. After adaptation to a larger numerosity, we cannot help underestimating subsequent stimuli, even if we are informed about their correct numerosity (Burr & Ross, 2008). The effects of adaptation are even more striking when different regions of the visual field are adapted to different numerosities. For example, if the left region of the visual field is adapted to a smaller numerosity, and the right region is adapted to a larger one (by staring at a mid-point between two clouds of dots, the smaller on the left and the larger on the right), subsequent numerosities presented on the left will be perceived as larger (overestimated), whereas subsequent numerosities presented on the right will be perceived as smaller (underestimated), even in the case that both numerosities are equal and the subject is informed about that. Dehaene (2009a, p. 233) comments on this result:

> the fact that numerical percepts impose themselves upon us so immediately, automatically, and without conscious control (even if we know that the numbers are equal) points to the operation of a special and largely automatic processing system. As noted by Burr and colleagues, "Just as we have a direct visual sense of the reddishness of half a dozen ripe cherries, so we do of their sixishness."

This similarity between the perception of color and the perception of numerosity is very illustrative of how numerosity differs from cardinality. Comparing *number* with color, Frege writes: "while I am not in a position, simply by thinking of it differently, to alter the colour or hardness of a thing in the slightest, I am able to think of the Iliad either as one poem, or as 24 Books, or as some large Number of verses" (Frege, 1960, §22, p. 28). What Dehaene is pointing out in the above passage is that, when it comes to *numerosity*, we are just as unable to change it, "simply by thinking of it differently," as we are unable to change color, especially when stimulus is presented quickly.[8] This marks a fundamental difference

[8] If sufficient time is provided, we can change our perception of numerosity by shifting the attentional focus.

between perception of numerosity and assessment of cardinality.

Numerosity is perceived *as if* it were a mind-independent property of the environment because the detection of numerosity is conducted by automatic processes that are out of our active control. But numerosity *is not* a genuine mind-independent property of the environment, since in principle our perceptual system could parse stimuli differently, resulting in different numerosity evaluations of the same physical phenomenon. If it is the perceptual system that imposes the segmentation of stimuli into objects and the grouping of them into collections, then numerosity is a property of these internally concocted collections.

This does not mean, however, that numerosity is a purely psychological construct. Some studies have shown that objective characteristics of the input affect our perception of numerosity. For example, Fornaciai et al. (2016) and Burr et al. (2017) cite studies that show that the presence of items connected to each other in stimuli decreases the perception of numerosity. In these studies, experimenters presented participants with clouds of dots in which some pairs of dots were connected by a line. This modification inclined participants to identify connected pairs of dots as a single unit. "At modest numerosities, connecting 40% of dots led to a 30% reduction in apparent number" (Burr et al., 2017, p. 7). This shows that the segmentation of stimuli into objects is not arbitrary; rather, it relies on genuine properties of the environment.

Numerosity is an internally built psychological property, but one that is responsive to external physical properties of the environment. Numerosity seems to emerge in the interplay between subjects and the environment. If this is so, numerosity perception bears more similarities with color perception than merely the fact that both are not subject to deliberate modification by the perceiver. At this point, it is worth briefly digressing into color perception, so that we can draw a closer parallel between it and numerosity perception. For this digression, I rely mostly on Chirimuuta (2015).

Although our everyday experience suggests that color is a property of objects we see in the external world, there are at least two sources of concern regarding this naïve view. First, color is subject to perceptual variation: depending on the angle of view, background, and lighting conditions, among other factors, the perceived color changes, and there is no principled way of determining which is the "real" color of the object in question. The following famous passage by Russell illustrates this point:

> Although I believe that the table is "really" of the same colour all over, the parts that reflect the light look much brighter than the other parts, and some parts look white because of reflected light. I know that, if I move, the parts that reflect the light will be different, so that the apparent distribution of colours on the table will change. … It is evident from what we have found, that there is no colour which pre-eminently appears to be *the* colour of the table, or even of any one particular part of the table—it appears to be of different colours from different points of view, and there is no reason for regarding some of these as more really its colour than others (Russell, 1951, p. 8-9).

Variations in perceived color, along with the lack of a principled way to select the "correct" variation, suggest that there may be an unavoidable mismatch between the colors we

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For example, regarding two connected dots, initially subitized as one single object, we can shift the attention to the dots only, and then subitize two items.

perceive from our personal standpoint and the underlying reality where objects could have a real color, inaccessible to us. However, the second source of concern for naïve realism about colors threatens precisely the idea that objects are, in reality, intrinsically colored. In the scientific description of the world, the fundamental constituents of matter—atoms and subatomic particles—are colorless; color does not appear among the properties of the world as described by physics (Chirimuuta, 2015, section 1.2.2). Russell's brown wooden table is intrinsically colorless, since it is made out of colorless atoms. Taken together, variation of color perception and the physical description of the world suggest that color is an illusion, an attribute our perceptual system projects onto reality. But this is not the whole story.

Against the anti-realism suggested by the above considerations, realists about color claim that color perception amounts to the detection of a property of surfaces called spectral surface reflectance (SSR). Roughly, SSR has to do with how a given surface absorbs and reflects different light wavelengths. Anti-realists, however, retort that light perception is not fully reducible to SSR. One problem is that SSR is a property that a surface has independent of the objects neighboring it, whereas the perception of the color of a given surface varies according to the objects around it. For example, a gray surface will be perceived as darker if surrounded by lighter gray surfaces, whereas it will be perceived as lighter if surrounded by darker surfaces (Chirimuuta, 2015, section 3.3.2).

In light of these problems, some have suggested that color is a relational property, i.e., a property of the relation between perceivers and their environment. Chirimuuta endorses this view. According to her, "colors are not properties of things (minds or extra-dermal objects) but of specific kinds of events, namely perceptual interactions" (Chirimuuta, 2015, section 6.2, para. 2). Color is not a property of the environment, nor a projection from subjects onto reality, but a phenomenon that emerges in the interplay between perceivers and the environment. Chirimuuta characterizes color as follows:

> Colors are properties of perceptual interactions involving a perceiver (P) endowed with a spectrally discriminating visual system (V) and a stimulus (S) with spectral contrast of the sort that can be exploited by V (Chirimuuta, 2015, section 6.2, para. 2).

In line with the realist view, in Chirimuuta's definition, the objective part of color perception, (S), also comes from the external world. It is composed of the SSR of surfaces and other external factors that influence color perception. But these external factors are not everything. The perceiver's perceptual system elaborates on top of (S) in order to produce the full experience of color perception. In Chirimuuta's words, "[c]olors are ways stimuli appear to certain kinds of individuals" (Chirimuuta, 2015, section 6.2, para. 7).

Now, the parallel between color and numerosity perception can easily be drawn. As we saw, numerosity is not a property of the environment because there is no characteristic manner in which physical aggregates can be parsed into discrete elements and grouped into collections. At the same time, perception of numerosity is not fully arbitrary, since there are some genuine properties of the external world that influence numerosity perception (e.g., connectedness). Thus, full-blown anti-realism about numerosities is not satisfactory, since numerosity perception does reflect aspects of the external world. But realism is not satisfactory either, since numerosity perception depends on how our perceptual system individuates and groups objects. Thus, a version of the midway relationist position about color perception seems to fit nicely here. Paraphrasing Chirimuuta's relationist characterization of colors, we can characterize numerosity as follows:

> Numerosity is a property of perceptual interactions involving a perceiver (P) endowed with quantical cognition (Q) and a stimulus (S) liable to segmentation into discrete objects of the sort that can be exploited by Q.

Numerosity is a property of perceptual interactions because it may vary across perceptual events and across individuals, just like perception of colors. Under the effect of adaptation, for example, perception of numerosity may vary across trials within the same individual. These variations cannot be regarded as mere "noise," as if they were the result of a kind of malfunctioning of our cognitive system that would prevent us from having clear sight of the discrete quantities before us, simply because there are no discrete quantities before us, given Frege's argument. Numerosity emerges only in the course of a perceptual interaction, when the perceiver receives the input stimulus and elaborates on it. This elaboration aims at ascribing a sense of magnitude to the stimulus as a function of the number of discrete items the agent's perceptual system identifies and collects in it.

For the perceptual interactions that give rise to numerosity perception, the quality of the stimulus (S) is not irrelevant. As specified in the characterization of numerosity given above, the stimulus must be liable to segmentation into discrete objects. For example, if the stimulus is completely homogeneous, like a continuous tone or a blank screen, there are no discontinuities—e.g., pauses, contrasting colors, shapes—which could be exploited for its segmentation into discrete items, and therefore perception of numerosity will not take place. It has been experimentally demonstrated that stimuli where objects are densely packed—where items become "crowded"—do not elicit numerosity perception (Anobile et al., 2016; Burr et al., 2017). Another indication that features of the stimulus significantly influence numerosity perception is the fact that the spatial organization of elements in a scene determines how collections will be formed. I mentioned above that whether items are connected or not makes a difference in numerosity perception. Gestalt psychologists have identified a number of other principles that govern the segmentation and grouping of objects in the visual field (Wagemans et al., 2012). One of these is proximity, according to which items that are closer to each other than to other elements of the scene tend to be grouped together. For example, compare the following strings of stars:

★★★★★

★★ ★★ ★★

In the top row, we tend to group all the stars together, whereas in the bottom row we tend to see three groups of two stars each. Proximity is a relation that objects have amongst themselves, independent of the perceiver. Again, this is clear evidence that numerosity perception is not detached from properties of the external world, although we cannot regard numerosity itself as a property of the external world. Again, it is a property of perceptual interactions.

Metaphysically speaking, we might say that perception of numerosity is only possible because we live in a world that produces stimuli in such a way that our perceptual system can segment them into discrete objects and group these objects into collections. In a sense, perception of numerosity is possible because we live in a world that "affords" segregation and collection (Gibson, 2015; Jones, 2018). If we lived in a liquid world, where there were no clear and enduring borders between objects, or in a world where ephemeral objects kept popping out and vanishing away all the time (like rain drops permanently floating in the air and fusing with each other or breaking apart continually) perception of numerosity would most likely not emerge.

At this point, the differences and similarities between numerosity and cardinality should be clear. Numerosity is a property of perceptual interactions. It can be thought of as a layer of information that our cognitive system adds to raw stimuli. Cardinality, on the other hand, is a property of collections of items, perceived or not, which we freely create. Cardinality is not a property of perceptual interactions—since we cannot apprehend cardinality simply by sight or hearing—but it is also a relational property: just as numerosity, cardinality emerges only after a cognitive agent has individuated and collected items. To assess cardinality, we do not simply interact perceptually with the collection thus created, but rather execute a procedure over its items, i.e., we count them. Since cardinality and numerosity emerge through different relationships—different ways of interacting with the environment—they correspond to two different relational properties. Figure 3.10 illustrates this point. Numerosity and cardinality may coincide when we deliberately count collections of up to three or four objects following the instinctive way our perceptual system parses the environment. Above this limit, however, perception of numerosity and evaluation of cardinality tend to diverge, even if, for the assessment of cardinality, we individuate objects in the instinctive way our perceptual system does. This marks another difference between numerosity and cardinality: the latter is a clear-cut property, whereas the former is vague.

In light of the conceptual distinctions I am drawing here, a cognitive scientist probing the quantical skills of a monkey can say, using the terminology proposed here, that the monkey perceives collections with different cardinalities as having the same numerosity (provided, of course, that the experimenter is parsing the stimulus using the same "sortal" used by the monkey's perceptual system). This is different from saying that the monkey perceives collections with different cardinalities as having the same *cardinality*. If the monkey is not able to count, the relational property "cardinality" never appears to it.

Numerosity is perceivable at a glance. Cardinality, by contrast, is not; its determination requires the execution of a deliberate sequence of steps: one has to deliberately choose a sortal, individuate objects according to this sortal, group them into a collection, and count the collection. Assessments of cardinality above the subitizing range require mastery of a technique. Non-human animals and innumerate humans cannot determine the cardinality of collections with more than three or four items, since they have not mastered the relevant technique.[9] The distinction Frege draws, in the following passage, between color and "Number" gives the lines of the distinction we should draw between numerosity and cardinality:

> It marks, therefore, an important difference between colour and Number, that a colour such as blue belongs to a surface independently of any choice of ours. … The Number

[9] The distinction made here between numerosity and cardinality is similar to the distinction that the psychophysicist S. S. Stevens proposed more than eighty years ago (Stevens, 1939/2006). The main difference is that Stevens uses the term 'numerousness' where I use 'numerosity,' and 'numerosity' where I use 'cardinality.'

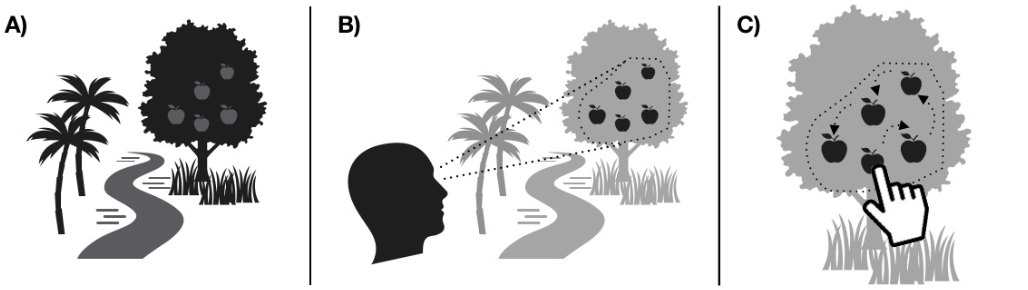

Figure 3.10: In (A) there is no observer, and therefore no numerosity or cardinality. Surely, the reader is able to see two palm trees or three trees, a few fruits, etc. In this case, though, an observer is present (the reader), which corresponds to the situation illustrated in (B). In (B), an observer focuses her gaze on the fruits, groups them into a collection, and thus perceives a certain numerosity. The emerging relational property is numerosity since she simply looks at the fruits, without counting them; this property emerges through a *perceptual* interaction handled by quantical cognition. In (C), the observer focuses her attention on the fruits and counts them. The emerging relational property, in this case, is cardinality, since it emerges through a *procedural* interaction in which the agent pairs each fruit with a counting word in the conventional fashion. This procedural interaction is handled by numerical (as distinct from quantical) cognition. This explains why numerosity is not "noisy" or "approximate" cardinality: without the participation of an observer, there is no cardinality out there to be seen (A); when the observer focuses on certain features of the stimulus, then numerosity (a vague sense of discrete magnitudes) emerges (B); only later, if the observer is able to count, can she follow a rule-based procedure with the features of the stimulus that elicited numerosity perception and thus determine cardinality (an exact assessment of discrete magnitudes) (C). Roughly, cardinality is better described as "exact numerosity" than numerosity as "approximate cardinality."

> 1, on the other hand, or 100 or any other Number, cannot be said to belong to the pile of playing cards in its own right, but at most to belong to it in view of the way in which we have chosen to regard it (Frege, 1960, §22, p. 29).

'Choice' is the key word here. Numerosity belongs to a stimulus, by and large, regardless of our choice. The mechanisms of our perceptual and cognitive systems that determine numerosity are mostly out of our active control, whereas the techniques we use to evaluate cardinality are mostly under our active control. This might reflect the fact that the cognitive mechanisms underlying numerosity perception have genetic roots—i.e., are "instinctive"—whereas the cognitive mechanisms underlying our understanding of cardinalities and cardinal numbers—our favorite yardstick for cardinality—are acquired only after extensive training.

All things considered, a concise definition of numerosity may run as follows:

> Numerosity is a property of perceptual interactions, evaluated by means of quantical skills, that refers to the perceived cardinal magnitude of the stimulus as a function of the sortal used by the agent's perceptual system to identify and collect discrete items in the stimulus.

Notice that the "sortal" here is not necessarily linguistic. Pre- and nonverbal agents will not individuate objects based on linguistic concepts, but can do so by relying on perceivable characteristics of the stimulus. For example, Feigenson and Halberda (2008) and Moher, Tuerk, and Feigenson (2012) have shown that infants can individuate and group objects in different manners guided by spatial and featural cues of visual stimuli.

3.6 A first hint on the nature of numbers

Quantical cognition is non-numerical because it neither allows those who possess it to see numbers in the environment nor relies on numbers to assess numerosity. Although quantical cognition is non-numerical, the discussion above can shed some light on the nature of numbers. First, we have seen that there is nothing in the findings from numerical cognition that contradicts Frege's claim that number (and cardinality) is not a property of the environment. Rather, Frege's observation helps illuminate what is really perceived through quantical abilities. Again in line with Frege's observations, we have seen that the internal mechanisms of quantical cognition do not make use of numbers—there are no numbers in the brain—nor representations of numbers. Given that innate number concepts have not been reported in other cognitive systems, we can conclude that number concepts are not innate. In sum, numbers are not to be found in the environment nor in the mind as innate cognitive resources.

These are negative conclusions: they say what numbers are not and where they are not to be found. However, from them we can also gain a more positive insight into the nature of numbers. Since number concepts are not a product of genetic evolution, everything we know about numbers, from counting to formal arithmetic, has to be learned from other human beings and, therefore, is a product of socio-cultural processes. This is in line with the hypothesis that numbers are human creations. However, the evidence gathered so far, from studies on quantical cognition only, is not sufficient to make a convincing case for this. We still need to investigate the way human beings acquire number concepts and how number concepts have emerged in history. I address these issues in the next two chapters.

3.7 Conclusion

Conceptual clarity is key to a well-informed philosophical account of any topic. More often than not, scientists do not use philosophically relevant terms carefully. This is exactly what we have seen in the literature on non-symbolic numerical cognition. In this conclusion, it is worth summarizing the most significant definitions and distinctions I have put forward in this chapter.

We saw that scientists usually mix up the concepts of cardinality, number, and numerosity. This confusion underlies the claim that quantical cognition is numerical. A crucial difference between cardinality and numerosity on the one hand, and numbers on the other, is that the former are properties, whereas the latter provide a method for evaluating one of these properties (cardinality). There are other, non-numerical ways of evaluating the size of discrete magnitudes. Cardinality can be assessed by one-to-one correspondence, in which case no numbers are involved. The same goes for numerosity. The mechanisms that implement subitizing assess the numerosity of small collections by one-to-one correspondence. The mechanisms that implement the perception of larger numerosities also rely on non-numerical means, which may be either one-to-one correspondence (in the linear model of the ANS) or an imprecise correspondence between elements of the to-be-enumerated collection and intervals of a continuous "stream" (in the logarithmic model). Thus, both subitizing and estimation rely on non-numerical means for the assessment of numerosity.

Both numerosity and cardinality are relational properties, i.e, properties that emerge in the relationship between cognitive agents and their environment. However, a crucial difference between numerosity and cardinality lies in the means through which they emerge: numerosity emerges in perceptual interactions handled by quantical cognition, whereas cardinality emerges in procedural interactions (counting) handled by numerical cognition. In other words, numerosity differs from cardinality in that the latter is not a perceptual property—it cannot be evaluated at a glance. Although numerosity is perceived as if it were a property of the external world, it is not. It emerges only when a perceiver, equipped with quantical skills, receives an input stimulus with certain properties. The perceiver's perceptual system exploits objective properties of the stimulus to parse it into discrete items, which are subsequently grouped in collections, which then have their numerosity assessed. Numerosity is a layer of information that our cognitive system adds to raw stimuli.

The observation that "number" is not a property of the environment nor an innate cognitive resource suggests that numerical competence is a product of cultural processes. The cultural processes underlying number learning are the topic of the next chapter.

Chapter 4

Numerical cognition

NUMBER concepts are not innate, as we saw in the previous chapter. If we are not born with number concepts, we have to acquire them somehow. We also saw in Chapter 3 that there are no numbers to be perceived in the environment, and thus the acquisition of number concepts cannot be like the acquisition of other concepts such as *dog*, *cat*, or *table*, which are acquired by means of direct contact with the things they refer to. But then, where do number concepts come from? In Chapter 2, based on a brief review of some findings from numerical cognition, I suggested the hypothesis that we acquire number concepts by initially experiencing numerals as de-semanticized symbols governed by operational rules. In other words, the suggestion is that numerals themselves, in conjunction with the practices wherein they are involved, such as the counting procedure, engender in us their meanings. In this chapter, I discuss in more detail results from numerical cognition and developmental psychology that give further support to this hypothesis.

I start this chapter with a preliminary discussion, in section 4.1, about what it means to possess number concepts. In section 4.2, I review correlational studies which show that, although quantical and numerical cognition seem to be positively correlated, the contribution of quantical skills to the development of numerical skills is modest. Exposure to number talk and experience with numerical symbols seem to be much more relevant for the development of numerical competence than quantical skills. In section 4.3, I review studies in developmental psychology that investigate how children learn to count. These studies show that first, children learn the sequence of counting words by rote, without associating numerical meanings to the words. Only later, when they have already mastered the counting principles, do they start to associate the right cardinal value with each word, clearly indicating that the acquisition of number concepts (for numbers larger than three or four, at least) is a consequence of learning to count. These results offer strong support to the hypothesis that we acquire number concepts by initially experiencing numerals as de-semanticized symbols governed by operational rules. In sections 4.4 and 4.5, I review the two most widely accepted accounts of the internal processes that take place when children build number concepts. In both accounts, the process of the formation of number concepts is triggered by the acquisition of number words and starts with collaboration from quantical skills. These two competing approaches disagree, however, about the importance of each of the quantical cognitive systems (ANS and OFS) in this process. As we will see, this debate is still ongoing. In section 4.6, I review results that show that mental cardinal values bear marks of the particular symbolic system through which they were learned, lending additional support to the thesis that numeral systems, conjointly with the counting procedure, give rise to number concepts. In section 4.7, I explain why counting is a cognitive tool indispensable for the acquisition of concepts for numbers larger than three or four.

4.1 Number concepts

As already noted in Chapter 2, I am using the term 'concept' to refer to psychological entities and this should not be taken as implying that *numbers* are mental entities. A concept, understood as a mental entity, may still be about external non-mental entities. In this chapter, I will not address the question of what number concepts are about. Rather, I will be concerned only with the ontogenetic (i.e., regarding the developmental history of an individual) origins of number concepts.

People's number concepts may vary considerably. This variation, however, is not necessarily in quality, but in quantity. Some people know much more about numbers than others. A mathematician can spend her whole life investigating numbers. As a result, her concept of number (i.e., her mental contents about numbers) will be very vast. However, we ascribe numerical understanding to people who know much less about numbers than mathematicians do. What is the minimum level of understanding of numbers that one must display in order to count as having some numerical competence?

In numerical cognition studies, the minimum requirement for a child being classified as a "number-knower" at some level is to pass the so-called Give-a-Number test (Wynn, 1990).[1] In this test, a certain number of objects—e.g., toy dinosaurs—is made available in a large bowl placed near the child being tested. Then the experimenter asks: "Could you give me one dinosaur?" If the child succeeds in giving exactly one dinosaur and does not give the same quantity when another number is asked for, she is said to be a "one-knower." Next, the experimenter asks: "Could you give me two dinosaurs?" If she succeeds, giving exactly two dinosaurs and not giving the same quantity when another number is asked for, she is a "two-knower." This process is repeated for the next numbers. As we will see below, children go through these levels, from one-knower to three- or four-knower, step by step, until they have finally learned to count arbitrarily larger numbers. This stepwise progress shows that it is possible to be familiar with the number one only, and after that with one and two only, and then with one, two and three only, before one has mastered a general method of counting. A similar observation can be made about people in few-number cultures.[2] As we will see in Chapter 5, in cultures in which the upper limit of the numeral system is very low (four or less), people do not know a general method for counting, but even so they are able to consistently and accurately use their number words for one, two, and three.

Following this convention established in developmental studies, I will assume that the

[1] A number-knower is not necessarily one who has justified true beliefs about numbers. A number-knower simply is one who knows the meaning of number words.

[2] Few-number cultures are small-scale cultures whose language has only a small list of number words (De Cruz, Neth, & Schlimm, 2010).

minimum level of number understanding is "one-knower." In other words, number understanding starts with the comprehension of "one" as referring to an exact quantity comprising a single item. This convention is far from being arbitrary: both cognitively and logically, the number one can be seen as the basis from which children generate all the subsequent number concepts (Buijsman, 2019).

If people may know only some numbers, then we have to distinguish between the concepts of each number—ONE, TWO, THREE, etc.—and the concept of NUMBER or, more precisely, NATURAL NUMBER. The concept of natural number involves the idea that the sequence of numbers is infinite. In mathematics, the standard definition of natural number is given through the Peano Axioms and only an infinity sequence can satisfy them. In mathematics, each number also has its own definition. If the concepts of one and successor are taken as primitive, then two is defined as the successor of one, three as the successor of two, and so on. Consequently, in mathematical terms, knowing the concept of, say, two, entails knowing all the numbers, since in order to know two one has to know the successor function. However, this is not how people really learn numbers in practice. As indicated above, children first learn the first numbers of the sequence individually, and only much later do they generalize the successor function and then realize that the sequence of numbers is infinite. In cognitive terms, then, having the concept of n does not entail having the concept of the successor of n.

What must one know about a number n in order to count as having the concept of n? The number one has many properties—e.g., it is odd, it is the smallest positive number, etc.—but a one-knower is not required to be aware of these properties. In the Give-a-Number task, when the experimenter asks "Could you give me one dinosaur?", what is being observed is whether the child is able to produce a collection of things with cardinality one, i.e., it is sufficient for her to know the cardinal value of the word /one/. Thus, according to the Give-a-Number task parameter of success, the minimum one must know about a number n to count as having the concept of n is its cardinal value.

We saw in section 3.5.1 that the cardinality of a collection can be determined and expressed with or without numbers. Numbers provide a method of precisely assessing cardinality, but there is another, viz., one-to-one correspondence, which does not involve numbers. Based on this, I claimed there that the concept of cardinality is independent of the concept of number. Now, we see that, cognitively, number concepts are in fact dependent on the apprehension of cardinalities: the minimum content of a number concept is its cardinal value. A cardinal value can be conceived of as an "abstract cardinality," i.e., the idea of the size of a collection that is composed of items whose identities do not matter or have been omitted. In other words, someone who knows the cardinal value of n knows what a collection with n items should "look like" in terms of size.

The introduction of cardinal values illuminates what we gain by using numbers in comparison to one-to-one correspondence. By means of one-to-one correspondence, we can compare particular cardinalities only. We always need another particular collection—a *model collection*—to express the cardinality of the *target collection*. For example, if we want to know how many people there are in a room—the target collection—by means of non-numerical one-to-one correspondence, we need another collection to model the collection of people in the room. One option may be the collection of chairs in the room. If there is a one-to-one correspondence between people and chairs, then we know that (a) "there are as many people as chairs in the room." However, someone who is not familiar with this model collection (chairs) cannot have any idea about how big the group of people in the room is after being informed of (a). Are there hundreds of people in the room or only a small group? Without using numerical means, one can only by inspecting the model collection or the target collection directly have an idea of their sizes. In contrast, by using a numeral we can convey information about the size of these collections without needing to present them. We can simply say "the number of people in the room is three." When a three-knower hears or reads this sentence, she knows what the collection of people in the room "looks like" in terms of size. 'Three' elicits in her mind the idea of a collection consisting of distinct items *a*, *b*, and *c*. This idea plays the role of a model collection—it works as a mental "model collection"—with the advantage of being a shared way of referring to cardinalities, facilitating mutual understanding, since most people associate a similar representation with 'three'. The cardinal value of a number is like a "standard model collection" that represents a cardinality. In Chapter 5, we will see that a strategy like this—the selection of a paradigmatic model collection to represent a cardinality—is behind the origin of numbers words for one, two, and three in some languages.

Above, I said that cardinal values can be conceived of as *abstract* cardinalities. In philosophy, there are different ways of spelling out what abstractness is. One way, which is compatible with concepts being psychological entities, is the Aristotelian way, according to which an abstract idea is one that is obtained by considering several experiences and omitting the features that distinguish them. Cognitive scientists claim that number concepts are abstract in an Aristotelian sense. In the literature on numerical cognition, a number concept is said to be abstract if it is recruited regardless of the modality (e.g., visual or auditory) or code format (e.g., symbolic or iconic) of the stimuli (Campbell, 2015). Or, as Dehaene et al. (1998, p. 356) put it, "[a]dults can be said to rely on an abstract representation of number if their behavior depends only on the size of the numbers involved, not on the specific verbal or non-verbal means of denoting them." However, the abstractness of psychological number concepts in this sense is controversial. There is mounting evidence that there might be several distinct instances of the same numerical facts—and, possibly, of the same cardinal values—in the brain. These different instances are likely to be recruited depending on the modality and format of the stimuli, resulting in alterations in behavioral parameters such as accuracy and reaction time (Campbell, 2015; Kadosh & Walsh, 2009; Kutter et al., 2018). However, even if number concepts are not abstract in the sense of being independent of modality and code format, these modality- or format-specific instances of numerical facts are still abstract in a broader sense. For example, even if reading '3,' hearing /three/, and seeing ◊◊◊ activate different instances of the cardinal value of three, the fact that different episodes of hearing /three/ brings to the mind of the hearer the same cardinal value regardless of the context in which this symbol is heard, the collection which it refers to, etc., shows that it is still abstract with regard to context, identity of the objects in the collection, etc. This is the sense in which I am saying that cardinal values are abstract cardinalities.

The next sections flesh out the details of how the minimal content of the first number concepts—their cardinal values—are acquired by individuals.

4.2 The relationship between quantical and numerical cognition

Quantical cognition is non-numerical, as we saw in the previous chapter. This does not mean, however, that quantical and numerical cognition are not related. The prevailing view is that quantical cognition provides the genetically evolved preconditions for numerical cognition.[3] Perhaps the most often cited experimental result that supports this view is the observation of distance and size effects—the hallmarks of the ANS—in the comparison of numerals. This observation was first made by Moyer and Landauer (1967). They presented numerate adults with two Arabic digits and asked which one represented the larger value. The reaction time and error rate of participants increased as the numerical distance of the presented digits decreased. For example, the comparison '7' vs. '9' took longer and was more error prone than the comparison '3' vs. '9.' "Considering this evidence," Holloway and Ansari (2015, p. 533) write, "it seems clear that the semantic referents of numerical symbols are built upon the approximate representations of numerical magnitude seen in non-human animals and illiterate humans."

The relation between quantical and numerical cognition is hardly deniable. However, the contents that numerals activate in the mind cannot be fully equated to numerosity representations for at least two reasons. First, distance and size effects are less intense in symbolic number comparison than in numerosity comparison (Verguts & Fias, 2004). Second, the neuronal populations that encode numerosity representations and number concepts barely overlap. In a recent study, Kutter et al. (2018) recorded the activity of single neurons of neurosurgical patients who had intracranial electrodes implanted in their medial temporal lobe (MTL) while they performed calculation tasks with numerosities or Arabic digits. The authors observed that neurons responsive to numerosities or Arabic digits "are largely segregated in the MTL; abstract neurons that encode the same numerical value in both non-symbolic and symbolic formats were rarely found" (Kutter et al., 2018, p. 756). However, they also observed that the activity of neurons exclusively tuned to Arabic digits displayed the distance effect too, though it was less intense than the effect displayed by numerosity-tuned neurons, as predicted by behavioral studies.

Taken together, these two observations—separate neuronal populations both displaying the distance effect—suggest that either number concepts are the product of the refinement and specialization of neurons previously responsive to numerosities; or they constitute an entirely new web of mental contents built under the influence of numerosity perception, but not exactly *upon* it. Roughly, these two hypotheses correspond, respectively, to Dehaene and Carey's accounts of the ontogenetic development of numerical cognition I address below. For now, the important message is that, in either case, numerosity representations and number concepts are clearly distinguishable in the brain, although related in some way.

Correlational studies have investigated the connection between quantical skills and numerical competence. Usually, in these studies, children have the accuracy of their quantical

[3]Even the nativist would agree with this view if 'quantical cognition' were replaced with 'non-symbolic numerical cognition.' The contentious point is how much of symbolic numerical cognition is already present in non-symbolic numerical/quantical cognition. Those who prefer the term 'quantical cognition' hold that the answer to this question is "very little." Nativists, in contrast, believe that number concepts are already in place at birth, and thus children only have to learn the right symbols to express the concepts they already have.

abilities measured and statistically compared to their performance on standardized arithmetic tests. In one of the pioneering studies of this kind, Halberda, Mazzocco, and Feigenson (2008) showed that the ANS accuracy of adolescents retrospectively predicted their performance on arithmetic tests from as early as kindergarten. In their study, students with a smaller Weber fraction (i.e., with a more accurate ANS) at the age of 14 achieved better scores in standardized arithmetic tests administered annually in their previous school years. However, such a significant correlation between quantical cognition and arithmetic skills was not confirmed in subsequent studies (e.g, Holloway and Ansari (2009); Lyons, Price, Vaessen, Blomert, and Ansari (2014); Sasanguie, Defever, and Reynvoet (2014)).

In a review of studies of this kind, De Smedt, Noël, Gilmore, and Ansari (2013) analyzed 11 studies that found a significant correlation between ANS accuracy and mathematical competence, and 14 that did not. They propose that differences in the sample size, the age of the participants, and the tests used to measure both quantical and numerical skills may at least partially explain these conflicting results. Despite methodological issues, however, they point out that

> the difficulty in finding relationships between non-symbolic numerical magnitude processing and mathematics achievement may indicate that the kinds of representations and processes measured by these tasks are not particularly critical for children's development of school-relevant mathematical competencies (De Smedt et al., 2013, p. 54).

In other words, the explanation for these conflicting results may lie in the fact that the influence of the abilities to estimate and to perform non-symbolic calculations on the acquisition of arithmetic competence is weak. De Smedt and colleagues' observations were confirmed by Schneider et al. (2016), who conducted a meta-analysis of 45 correlational studies of this kind. Overall, they found only a weak correlation between quantical skills and arithmetical competence. More recent studies continue to provide mixed evidence. For example, Elliott, Feigenson, Halberda, and Libertus (2019) found a significant correlation between ANS accuracy and arithmetic abilities, whereas Hyde, Simon, Berteletti, and Mou (2017) did not. Bulthé, Smedt, and Beeck (2018) found that arithmetic skills correlate negatively with the overlap of numerosity representations and number concepts in the brain. In other words, "individuals with higher arithmetic skills have weaker associations between symbolic and non-symbolic representations compared to individuals with lower arithmetic skills" (Bulthé et al., 2018, p. 307).

The relation between the acquisition of arithmetical competence and the OFS has not received the same attention as the relation between the former and the ANS, but the few studies available on this topic also found conflicting results. Hyde et al. (2017) measured the spontaneous neural activity of the OFS when children were presented with small numerosities and tested with standardized arithmetic tests to evaluate their numerical competence. They found a significant positive correlation between OFS neural activity and numerical competence. By contrast, Anobile, Arrighi, and Burr (2019) took a different approach and found a different result. They measured children's subitizing limit in the conventional fashion, i.e., by presenting them with arrays of dots and asking how many dots they see, and compared this to their performance on standardized arithmetic tests. Their conclusion was that "subitizing limits do not correlate with mental calculation or digit magnitude knowledge proficiency … subitizing does not seem to be related to numerical abilities" (Anobile et al., 2019, p. 86).

All things considered, these conflicting results bring a twofold message: the influence of quantical abilities on the acquisition of arithmetic competence cannot be dismissed, but it seems to be only a small part of the story. There must be other factors whose contribution to the acquisition of number concepts and arithmetic competence is more decisive.

In fact, many studies that did not find a significant association between arithmetic skills and quantical cognition, did find a stronger correlation between the former and performance in symbolic magnitude comparison. A typical symbolic magnitude comparison task consists of presenting participants with two digits or two number words and asking which has the larger numerical value. In Schneider's et al. (2016) meta-analysis, the correlation between performance on arithmetic tests and performance on the symbolic comparison task was more significant than the correlation between the former and ANS accuracy. In De Smedt's et al. (2013) review, 13 (76%) out of the 17 reviewed studies that investigated this relation found a significant correlation between symbolic magnitude comparison and arithmetic competence. Bulthé et al. (2018, p. 307), who found a negative correlation between arithmetic skills and quantical cognition, suggest that

> [i]ndividuals with more skills and experience in arithmetic symbols might have symbolic representations [number concepts] that are more defined in terms of their relations with other numerical symbols than in terms of a reference to a fixed, concrete or perceptual quantity [numerosity representations].

What these results show is that a solid understanding of the symbolic system of numerals can boost the acquisition of more complex numerical competences. True enough, this is to be expected, since familiarity with numerical symbols is a prerequisite to perform calculations with symbols. But the puzzle lies in how children acquire the meaning of numerals in the first place, since it seems not to be significantly driven by inborn quantical abilities.

Some studies have shown that exposure to numerals themselves being used in everyday situations seems to play a major role in the acquisition of number concepts. Levine, Suriyakham, Rowe, Huttenlocher, and Gunderson (2010) observed several episodes of spontaneous interactions between children and their caregivers during a period of 16 months, and counted how frequently number words appeared in these interactions. Another 16 months later, when children were three years and ten months old, they assessed children's knowledge of the cardinal values of number words. They concluded that variation in the amount of number talk in the observed period was strongly correlated with children's number knowledge by this age. In a follow-up study, Gunderson and Levine (2011) investigated which types of number talk are most significant for children's number learning. They found that "number talk in which parents either count or label perceptually present sets of objects is more related to children's development of cardinal-number knowledge than number talk that does not," and especially "talk about sets that fall outside the range governed by the small-exact-number system (i.e., sets of size 4 to 10) is the strongest predictor of children's later cardinal-number knowledge" (Gunderson & Levine, 2011, p. 1030). Among the types of number talk strongly correlated with children's number learning are those present in situations where parents count collections of objects for children, explicitly showing how the counting process works. The authors give as an example a situation in which a parent counts blocks with pictures of hats on them while saying "[w]e got one, two, three, four, five, six, seven, eight more hats" (Gunderson & Levine, 2011, p. 1025).

In the same vein, Ramani, Rowe, Eason, and Leech (2015) showed that the frequency with which caregivers engage in activities involving numbers with their children correlates positively with children's numerical knowledge. Among the activities with the highest impact, they identified proper teaching activities, where caregivers directly teach children to count, and situations in which children engage with their caregivers in daily activities where numbers are used, such as measuring ingredients while cooking or timing an activity. Purpura and Reid (2016) showed that three-to-five-year-olds from families where both parents have a lower level of education lag behind their peers from families where at least one parent has a higher level of education in mathematical language proficiency, suggesting that "parents with higher levels of education more often provide richer home numeracy environments for their children" (Purpura & Reid, 2016, p. 265). In an intervention study, Niklas, Cohrssen, and Tayler (2015) showed that enhancement of the "home numeracy environment" improves four-year-old children's numerical abilities. Intervention measures included instructing parents on how to boost children's understanding of the counting principles and introducing a dice game.

Altogether, these results indicate that a process of enculturation, rather than innate quantical skills, is likely to be the main force driving children's development of numerical competence. The more parents and caregivers structure the environment in numerical terms (number talk, demonstrations of the counting method, daily activities involving numbers), the faster children acquire number concepts. In the next section, we will see what children must learn from their parents and caregivers in order to master the counting procedure and the steps through which they progress in the learning process.

4.3 Learning to count

In one of the pioneering major studies on the ontogenetic development of numerical competence, Gelman and Gallistel (1978) proposed a model of counting abilities that is still standard in numerical cognition. According to their model, in order to learn how to count, a child has to master the following principles:[4]

1. *One-to-one Correspondence*: each item of the counted collection must be paired with one and only one number word.
2. *Stable Order of Counting Words*: the order in which number words are used for tagging items must follow a stable order, i.e., the order must be kept constant across different

[4] Gelman and Gallistel's formulation of the counting principles is slightly different from the one given here. Instead of referring to number words, they refer to *tags*. In their formulation, tags may be either *numerlogs* or *numerons*. Number words and digits are numerlogs, whereas numerons are "any distinct and arbitrary tags that a mind (human or nonhuman) uses in enumerating a set of objects" (Gelman & Gallistel, 1978, p. 77). Thus, in their formulation, the counting principles are general enough to allow *nonverbal* counting. The assumption that non-verbal counting is possible is a consequence of their nativist view, according to which children are born with "numerons" already in place. Formulations of the counting principles in non-nativist terms, i.e., in terms of numbers words, are widely found in the literature (e.g., in Gilmore, Göbel, and Inglis (2018).

counting events.

3. *Order Irrelevance of Items*: the order in which items are paired with number words is irrelevant, i.e., the order may change across different counting events.

4. *Abstraction*: counting applies to any collection of all sorts of objects, including sets formed by physical objects or ideas, and heterogeneous sets, formed by a combination of different kinds of objects.

5. *Cardinality*: the number word used for tagging the last item of a collection represents the cardinality of the whole collection.

These five rules are known as the *counting principles*. They are the norms that regulate the cognitive practice of counting. The first four principles determine how to count. They summarize the procedural knowledge one must have in order to count. The last principle refers to the information one obtains by counting, namely, the cardinality of the counted collection. In order to be able to appreciate this information, one must know not only that the last number word refers to the cardinality of the whole collection, but also the cardinal value of that number word. Thus, learning to count requires the acquisition of new words, the command of a procedure, and the acquisition of cardinal values, *in this order*, as we will see next.

Learning to count starts with the acquisition of what is initially experienced as a *meaningless* sequence of words. It is well documented that children first learn the sequence "one, two, three, four, etc." by rote, without knowing what these words mean. "Until the age of about 2 years, infants will predominantly sing-song numbers without attaching any meaning to them" (Knops, 2020, p. 116). At about three years of age, typically developing children are able to recite an initial segment of the sequence of counting words in the correct order and start to display satisfactory control of the first four counting principles. However, they still lack conceptual understanding of number words and the counting routine. "In fact, over several years during development, there may exist this gap between procedural knowledge (i.e. reciting the count words) and conceptual knowledge (i.e. the understanding of numeral meaning)" (Knops, 2020, p. 115). That is, children start learning to count by executing the counting procedure mechanically. Conceptual understanding comes much later.

Wynn (1990) demonstrated this gap during the first stages of learning to count. In Wynn's experiments, young children who were able to count up to five and could establish a one-to-one correspondence between number words from 'one' to 'five' and a collection of five items failed systematically in the Give-a-Number task. When asked to give two, three, or five items, they simply grabbed a handful. If asked to count the objects, they counted satisfactorily, showing command of the four procedural counting principles. However, when asked how many objects they had just counted, they did not answer with the last word they had used; they preferred to recount the set, clearly showing that they had not yet understood the cardinality principle.

Children start to understand that the last number word used in a counting episode refers to the cardinality of the whole collection in a piecemeal manner. First, around two to two-and-a-half years of age, they learn that 'one' refers to a collection consisting of only one object. At this point, when asked to give one object, they answer correctly, but when asked to give two or more, they still answer randomly by just grabbing a handful. This shows that they already know that the meaning of other number words contrasts with the meaning of 'one,' even though they still do not know the exact meaning of other number words. About four to five months later, they realize that 'two' refers to a collection consisting of two objects, and then they succeed at Give-a-One and Give-a-Two tasks, but still fail when asked to give three or more objects. Another four to five months elapse until they realize the meaning of 'three.' Another four months later they realize the meaning of 'four,' and then a major realization takes place: they finally understand the cardinality principle, and then become able to succeed at Give-a-Number tests for arbitrarily larger numbers up to the limit of their already-memorized list of number words (Gilmore et al., 2018; Le Corre, Van de Walle, Brannon, & Carey, 2006; Wynn, 1992b).

Le Corre et al. (2006) established the now standard classification of the milestones in the process of learning to count. After the earliest stage in which children simply sing-song number words, they start acquiring conceptual understanding of number words as *one-knowers* (when they know the meaning of /one/ only), and then they successively become *two-knowers* (know the meaning of /one/ and /two/), *three-knowers* (know the meaning of /one/, /two/, and /three/), and *four-knowers* (know the meaning of /one/, /two/, /three/, and /four/). Children at these stages are collectively called *subset-knowers*. Finally, they become *CP-knowers*, that is, they understand the cardinality principle and, as a result, can pass the Give-a-Number test for all the number words in the initial segment of the counting sequence they already know. For typically developing children in industrialized societies, it takes about one and a half years to go from one-knower to CP-knower. As Le Corre and colleagues point out, this prolonged process suggests that

> the acquisition of the verbal count list may involve the construction of a system of representation that is not innately available … children's representational resources undergo a drastic, qualitative change when they acquire the counting principles (Le Corre et al., 2006, p. 133).

If learning numbers were only a matter of mapping number words onto previously existing, innate concepts, once a child has realized that numerals refer to cardinal values—what happens once she becomes a one-knower; recall that one-knowers know that other numerals contrast with /one/—she would be expected to map all her memorized number words at once. But this does not happen. Children have to build the first number concepts one by one.[5] This, along with the observation that children start learning to count by reciting (to them) meaningless words within a rule-based procedure, constitutes strong evidence for the hypothesis advanced in Chapter 2 according to which number concepts originate from initially de-semanticized symbols whose use is regulated by operational rules.

Leaving nativists aside, the outline of the number learning process given above is a consensus among developmental psychologists and cognitive scientists. However, debate per-

[5]Nativists counter-argue that the observed extended time lapse between the acquisition of procedural and conceptual knowledge of counting may be the result of excessive performance demands on young children. For example, Cordes and Gelman (2005) argue that children's failure in Give-a-Number tasks is not a sign that they are not CP-knowers, but that they do not understand "how many" questions.

sists regarding how the ingredients of quantical cognition and numerical symbols and procedural command of the counting principles are combined in the construction of number concepts. Two major proposals have been advanced about this point. One, which I call the ANS-based account of number acquisition, holds that number concepts are built upon ANS representations of numerosities. The other, which I call the OFS-based account of number acquisition, holds that the OFS plays the central role in providing the first numerals with meaning, and that only much later numerals are mapped onto the ANS. In both accounts the role of numerals in shaping number concepts is decisive. Let us examine each account in turn.

4.4 The ANS-based account of number acquisition

The main proponent of the ANS-based account of the acquisition of number concepts is Stanislas Dehaene (e.g., Dehaene (2009a, 2011)). The gist of his account is the idea that children learn the cardinal meaning of number words by mapping them onto the ANS. Initially, this mapping might provide number words with an approximate meaning only. Over time, as children become proficient in using number words and counting, this mapping is believed to induce deep changes in innate numerosity representations, by means of neuronal recycling, giving rise to a new system of exact cardinal values.

The terminology with which the ANS-based account is presented is nativist: numerals are said to be mapped onto "approximate representations of numbers" that were in place since birth, which are said to be responsible for "non-symbolic numerical skills." As I have argued in section 3.5, though, this can be seen as a careless way of speaking; representations of numbers are not likely to be involved in Dehaene's model of the ANS. In fact, Dehaene's proposal is not genuinely nativist, since in his account, exact number concepts are not available before the acquisition of number words. Rather, it is the very acquisition of number words that sharpens ANS numerosity representations transforming them into exact cardinal values.

How such a transformation could take place has been demonstrated by Verguts and Fias (2004). They built a neural network to show that the association of symbols for numbers with numerosity representations has the potential to sharpen the latter. First, they trained the neural network to recognize numerosities by feeding it with collections of different cardinalities. At this stage, the network built numerosity representations quite similar to the ANS's, displaying distance and size effects. Then, they further trained the neural network by feeding it with collections paired with digits representing their exact cardinality. Then, they observed that the original numerosity representations became substantially more precise. The symbols had the effect of fine-tuning numerosity representations, so that the once ANS-like compressed "number" line gave rise to a linear number line with fixed variability, i.e., a system of representations where the difference between any neighbor quantity is constant, and thus more suitable for encoding cardinal values.

The process that leads from numerosity representations to cardinal values in children's brains may be similar. However, in contrast to what happened to Verguts and Fias's neural network, in the human brain the new number line with fixed variability does not replace the innate compressed "number" line. The two systems coexist throughout our life span. In an fMRI study with numerate adults, Piazza, Pinel, Bihan, and Dehaene (2007) showed that there are areas in the intraparietal cortex that respond to both digits and sets of dots—this indicates that a mapping of digits onto ANS representations really does take place—but there is also an area in the left intraparietal cortex that seems to be tuned exclusively to digits—these may be the neurons that were transformed and became specialized in encoding exact cardinal values. "[I]t is tempting to speculate that this sharpening and linearization effect only occurs in the left but not in the right parietal cortex" (Dehaene, 2009a, p. 252). More recently, as mentioned above, Kutter et al. (2018) found neuronal populations in the medial temporal lobe which are selectively tuned to either numerosity or number, which reinforces the claim that the two systems coexist. The pattern of activation Piazza et al. (2007) found in the left intraparietal cortex and Kutter et al. (2018) found in the neurons exclusively tuned to numbers is compatible with the linear number line with fixed variability in Verguts and Fias's (2004) model.

The transformation of numerosity representations into cardinal values in the left parietal cortex is an instance of neuronal recycling. As discussed in section 2.3, according to the neuronal recycling hypothesis, the learning of new cognitive skills takes place by recycling pre-existing brain circuitry that originally served a similar but simpler purpose. Because the recycling potential of neurons is limited, cultural acquisitions must take place within the bounded plasticity of their neuronal evolutionary precursors, and therefore the learned abilities still display features of their predecessors. This would explain why distance and size effects are also found in the comparison of numerals, as detected by Moyer and Landauer (1967) (Dehaene, 2005).

The ANS-based account predicts that children who start with a more precise ANS learn numbers faster. However, as mentioned above, evidence for this is mixed. Another of this model's predictions is that the acquisition of number words gradually enhances ANS accuracy. In a longitudinal study with three- to five-year-olds, Elliott et al. (2019) found a correlation between earlier numerical ability and later ANS precision after controlling for earlier ANS precision; that is, children experienced a significant enhancement in ANS precision after learning to count. This seems to confirm the effect of number words on sharpening numerosity representations. In another longitudinal study with three- to four-year-olds, Shusterman, Slusser, Halberda, and Odic (2016) investigated which stage of the initial development of numerical competence has the greatest correlation with enhancement in ANS accuracy. They observed that transitions across subset-knower levels were not accompanied by increments in ANS accuracy; only the acquisition of the cardinality principle coincided with improvement in the ANS. This observation calls into question the role of the ANS at the earliest stages of number learning.

In connection with this, Carey, Shusterman, Haward, and Distefano (2017) investigated whether subset-knowers learn the meaning of numbers words by mapping them onto the ANS. According to them, one idea that seems to be behind the ANS-based account is that, when children start to realize the meaning of number words—i.e., when they become one-knowers—they form the general hypothesis that number words map onto ANS representations. Then, to learn other number words, one thing they must find out is where each word falls in the ANS "number" line. The more often they see a number word being used in association with a certain numerosity, the faster they should infer where this word falls.

Carey et al. tested this claim by trying to interfere with this process. First, they taught three-knowers the meaning of 'four' by presenting them with several pairings of this word and collections of four objects. They observed that children quickly and robustly learned to apply the word 'four' to collections of four objects when contrasted with other set sizes. Then, they hypothesized that, if children had learned 'four' by mapping it onto the ANS, the same training strategy would work for teaching other number words, such as 'ten.' However, this hypothesis was not confirmed. They trained young CP-knowers to associate the word 'ten' with collections of 10 without counting, just by estimating, as adults can do. However, children learned to associate the word 'ten' only with the particular collections used in the training phase. The young CP-knowers failed when confronted with collections of 10 with other objects and configurations. They failed even to apply the word 'ten' to a collection of 10 when contrasted with a collection of 30, whose ratio is well above ANS resolution. Carey et al. concluded that this failure shows that children do not have an overall hypothesis that number words map onto the ANS. In the absence of this overall hypothesis, Carey and colleagues suggested that the children had learned the meaning of 'four' through other means, not relying on the ANS. In a previous study, Le Corre and Carey (2007) had already shown that young CP-Knowers fail to apply an appropriate number word to describe their estimates of collections with more than four items, even though they are able to count such collections. This, along with the fact that perception of small numerosities in the majority of cases recruits the OFS, rather than the ANS, suggests that the ANS plays a minor role, if any, in the acquisition of the first number words.

Even if the ANS does not play an important role in the beginning of the acquisition of number concepts, it is undeniable that, eventually, subjects map number words onto the ANS, otherwise they would not be able to express their estimations of numerosity in numbers. Izard and Dehaene (2008) provide a model of what the final configuration of this mapping may be. The controversy lies in *when* this mapping is established and *whether* it helps the acquisition of number concepts.

4.5 The OFS-based account of number acquisition

The main proponent of the OFS-based account of the acquisition of number concepts is Susan Carey (Carey, 2009). The gist of her account is the idea that children learn the meaning of the first number words by mapping them onto representations provided by a module that she calls "enriched parallel individuation." Enriched parallel individuation comprises the OFS (or "parallel individuation," as Carey calls it) and "set-based quantification," another innate system of prelinguistic quantity representation postulated by Carey.[6]

According to Carey, set-based quantification is responsible for providing meaning to the singular/plural distinction and to natural language quantifiers such as 'some,' 'each,' 'every,' and 'many.' Carey postulates the existence of this system because neither the OFS nor the ANS can supply representations for grounding the meaning of the singular/plural distinction and quantifiers. For example, the word 'some' represents a vague plurality that cannot

[6] Carey et al. (2017) propose another account of enriched parallel individuation in which set-based quantification plays no role. In this account, enriched parallel individuation is just the OFS enriched with the capacity to build long-term memory representations of collections of one to four items.

be mapped either to the exact numerosity representations of the OFS or to the approximate representations of the ANS ('some' is not approximately two, nor approximately three, and so on). Carey claims that the OFS and the ANS provide inputs to set-based quantification, which "regroups" them to produce representations that ground the meaning of the singular/plural distinction and quantifiers (Carey, 2009, p. 254ff.).

The role of the OFS in number learning is suggested by the discontinuity observed in the process of number acquisition, in which children learn the numbers that fall within the limit of the OFS (from one to three or four) in a manner markedly different from the way they learn the subsequent numbers, as we saw in section 4.3.

> If we accept that children in the subset-knower period do not know the significance of counting, it follows the cardinal content of the words "one" through "four" in the subset-knower cannot be provided by their role in a counting procedure constrained by the counting principles (e.g., the numeral "four" cannot receive its meaning by virtue of being the fourth word in the count list). This conclusion raises an important question: if the meaning of the first verbal numerals is not provided by their role in counting, how do they get their numerical content? (Carey et al., 2017, p. 244).

The OFS would be a natural candidate source of meaning for the first number words, if it were not for the fact that the OFS represents numerosities only in working memory and only implicitly (by one-to-one correspondence). Because the OFS does not provide enduring representations of numerosities onto which number words can be mapped, Carey proposes that these representations can be supplied by the interaction between the OFS and set-based quantification (Carey, 2009, p. 319-321).

There is some evidence from studies in developmental psychology that seemingly supports the claim that set-based quantification and the OFS cooperate in the acquisition of the first number words. Kouider, Halberda, Wood, and Carey (2006) showed that English-learning children start to realize the contrast between singular words in sentences such as "there is a cookie" and plural words in sentences such as "there are some cookies" around the age of two. This is the same age at which children begin to understand the meaning of 'one.' Based on this coincidence, Carey (2009, p. 321) hypothesizes that the meaning of 'one' is initially derived from the meaning of the singular marker 'a'/'an.' This hypothesis is further supported by studies that show that in languages without singular/plural markers, such as Japanese and Mandarin, children become one-knowers later relative to English speakers (Le Corre, Li, Huang, Jia, & Carey, 2016; Sarnecka, Kamenskaya, Yamana, Ogura, & Yudovina, 2007). In addition to this, there is the observation that when children learn the meaning of 'one,' they contrast it with all the other number words: when asked to give any greater-than-one number of items, they just grab a handful. According to Carey, one-knowers respond in this way because they see all number words after 'one' as meaning *some*. Thus, at this early stage, children supposedly conflate 'one' with the singular (a/an) and the other number words with the plural (some). The acquisition of the next numbers would come with familiarity with other linguistic quantifiers: the meaning of the word for two would be derived from dual markers, and for three from trial markers, in languages that have these markers.[7] Marušič et al. (2016) find that children learning a dual dialect of Slovenian learn the meaning of the word for two quicker than children learning a non-dual dialect

[7]English and many other modern European languages have only singular (one) and plural (more than one)

of the same language. In languages that do not have dual markers, like English, Carey claims that set-based quantification still plays a role in the acquisition of numbers from two to four, since cardinal numerals used in adjectival position (like in "there are two cookies") can be analyzed as quantifiers (Carey, 2009, p. 296).

Carey (2009, p. 324) emphasizes that set-based quantification does not have innate representations that can ground the meaning of singular/dual/trial/plural markers or the first numerals. These representations must be built in the course of language learning. The process starts in the OFS, where each item of an observed collection is ascribed to one of its memory slots. In this way, collections of one, two, and three items are represented in working memory as something like ⧫, ⧫⧫, and ⧫⧫⧫, respectively. These representations are non-numerical, as explained in Chapter 3. Set-based quantification receives such representations as inputs from the OFS in situations where singular/dual/trial/plural markers, quantifiers or numerals are heard. Over time, set-based quantification stores these inputs in long-term memory as the meaning of singular markers or 'one,' dual markers or 'two,' and trial markers or 'three,' respectively. Once these long-term memory representations are in place and associated with number words, in the presence of a new collection the child searches through them to find one that can be put in one-to-one correspondence with the OFS working memory representation of the collection under consideration, and in this way she selects the number word that applies to it.

What makes these long-term memory representations cardinal values is their association with number words. The set-based quantification's representation {⧫⧫⧫}, initially without any numerical content, becomes a cardinal value when it is stored as a paradigm of 'three' ("'three' means a collection like this"). Here I used a generic symbol (⧫) to depict enriched parallel individuation representations, which may suggest that these representations are abstract from the outset. However, as Carey (2009, p. 324) points out, "they could simply be long-term memory models of particular sets of individuals ({Mommy}, {Daddy Johnnie}...). What makes these models represent 'one' 'two' and so forth is their computational role." Carey et al. (2017, p. 246) give an illustrative example:

> Upon hearing this idea explained in a class, a colleague reported that for a couple of months, his 3-year-old daughter always commented on sets of three thus: "there are three kittens, mommy, daddy, me, three;" "Look, three cement mixers, mommy, daddy, me, three."

The learning strategy based on enriched parallel individuation reaches its limit when children become four- or three-knowers. From this phase onward, another process must account for the acquisition of subsequent number words, since for larger collections, the OFS cannot provide inputs to set-based quantification. According to Carey, in order to go on, children have to "bootstrap."

Bootstrapping is a metaphor used by psychologists "to explain learning of a particularly difficult sort—those cases in which the endpoint of the process transcends in some qualitative way the starting point" (Carey, 2004, p. 59). This is what happens with number learning in Carey's account. The starting point is provided by enriched parallel individuation, but it

markers. However, there are other languages, such as Slovenian and Arabic, which may distinguish between singular (one), dual (two), trial (three) and plural (more than two or three).

cannot go beyond three or four. The end point is the acquisition of the idea of a potentially infinite series of natural numbers. According to Carey (2009, p. 325ff), children bridge the gap between starting and end points by getting familiar with the list of number words as used in the counting procedure and by relying on a few other cues, as follows.

When children learn the words from ‘one’ to ‘four,’ they receive several cues regarding the meaning of the other number words. First, the use of plural words both for collections falling within and above the OFS limit suggests that all numbers words should be applied in a uniform manner. Second, the experiences children have with the first three or four number words might have already allowed them to grasp the essential procedural and semantic features of the counting process. They have already had the chance to observe that sets of two elements are counted “one, two;” sets of three are counted “one, two, three;” and sets of four are counted “one, two, three, four.” At least for these numbers, they may have already noticed the regularity with which the last word used in an instance of counting refers to the cardinality of the whole collection (i.e., they may have a partial understanding of the cardinality principle). Furthermore, they might have already understood that by adding one element to a collection of one, they obtain a collection of two; and by adding one element to a collection of two, they obtain a collection of three. Thus, for the first three or four words, they might have already noticed that the next word in the counting sequence means the cardinal value of the previous one plus one. Now, the only thing they need to do in order to become CP-knowers is to link this information to the uniform treatment of all number words suggested by plural markers. At this point, children may have all the elements they need to generalize this rule:

> if a word ‘x’ is followed by the word ‘y’ in the counting sequence, its meaning is the result of adding one to the cardinal value of‘ ‘x.’

Following this general rule, children can infer the meaning of ‘five’ from the meaning of ‘four,’ the meaning of ‘six’ from the meaning of ‘five,’ and so on, for all the number words in their already-memorized counting list. This is supposedly the “bootstrapping step.”

Notice that children’s previous knowledge of the numeral list as meaningless words is essential for bootstrapping. If they knew the numerals from ‘one’ to ‘four,’ but did not recite them along with other, still meaningless (for them) number words in the counting procedure, they would not have any reason to generalize. What invites them to generalize is the fact that they know that the sequence of number words starts with ‘one’ and continues after ‘four,’ and that the other number words in the list are used similarly to the words up to ‘four.’

In Carey’s account, neuronal recycling does not play any role. Number representations are not built upon previously existing numerosity representations. They are new mental contents, initially created with the aid of the OFS, and subsequently generalized to provide meaning to number words above four. This makes numerical competence an even more radical case of enculturation, in which the enculturated abilities exploit a previously existing cognitive system not to recycle it, but to build new contents.

Carey’s account has been shown to be computationally viable by Piantadosi, Tenenbaum, and Goodman (2012). They built a computational model that learns numbers by relying on a few primitive computational functions corresponding to the features of enriched parallel individuation, plus a probabilistic module responsible for statistical inferences. They fed the model with pairs of number words and collections of objects distributed over the frequency in which English-learning children are exposed to number words in real life. With this input,

> [t]he model … successively learns the meaning of "one", then "two", "three", finally transitioning to a CP-knower which correctly represents meaning of all number words. That is, with very little data the "best" hypothesis is one which looks like a 1-knower, and as more and more data is accumulated, the model transitions through subset-knowers. Eventually, the model accumulates enough evidence to justify the CP-knower lexicon that recursively defines all number words on the count list. At that point, the model exhibits a conceptual re-organization, changing to a hypothesis in which all number word meanings are defined recursively (Piantadosi et al., 2012, p. 207-208).

A study by Davidson, Eng, and Barner (2012) casts doubt on the nature of the generalization children make when they bootstrap. Their point is that the bootstrapping step can be decomposed into two generalizations: (i) the child generalizes the *procedural* rule according to which the last number used in a count is the answer to the question 'how many?'; and (ii) the child generalizes the *semantic* rule according to which the meaning of the next number word is the cardinal value of the previous one plus one. If this is so, it is possible for a child who has made only the first generalization to pass Give-a-Number tests by applying a mechanical procedure, without having really made a semantic generalization.

In order to test whether becoming a CP-knower really involves a semantic induction, Davidson et al. conducted two tests. In what they called the "more task," recently turned CP-knowers were presented with two boxes and asked: "This box has *N* stickers and this box has *M* stickers. Which box has more stickers?" Results for this task showed that children did not yet have full command of the cardinal values corresponding to number words. For comparisons as small as 'five' vs. 'six' and 'eight' vs. 'nine,' young CP knowers made correct judgments only 72% of the time. In the other test, which they called the "unit task," recently turned CP-knowers' performance was even worse. This task was intended to test children's understanding of the idea that the meaning of the next number word is the cardinal value of the previous one plus one. To this end, the experimenter put some beads in a box while saying "I am putting *N* beads in the box," and then asked the child: "How many beads are in the box?" After the child had given the right answer, the experimenter said "Good! Now watch!" and then added, one at a time, either one or two beads to the box. Then the experimenter asked: "Now are there *N+1* or *N+2* beads in the box?" (using the proper number words, not additions). Young CP-knowers' performance on this task did not differ from chance even for numbers as small as four and five.

It is possible that factors other than the absence of a semantic generalization in the bootstrapping step may have interfered with children's performance, but it is also possible that full command of the meaning of number words does not come immediately after the acquisition of the cardinality principle.

> Our results … suggest that the semantic induction may take place later. Therefore, at least on these grounds, it remains possible that the semantic induction is, in fact, driven by a mapping between the count list and the approximate number system (Davidson et al., 2012, p. 171).

The hypothesis that recently turned CP-knowers have just mastered a purely procedural rule, instead of making a semantic generalization, has been confirmed in several subsequent studies (Cheung, Rubenson, & Barner, 2017; Schneider et al., 2020; Spaepen, Gunderson, Gibson, Goldin-Meadow, & Levine, 2018; Wagner, Cheung, Kimura, & Barner, 2015). Carey herself has recently admitted that, differently from her original proposal, a semantic generalization is not what causes children to become CP-knowers. These new findings suggest that it is the other way around: becoming a "mechanical" CP-knower is a prerequisite for making a semantic generalization (Carey & Barner, 2019, p. 831). In other words, procedural command of the five counting principles comes before the acquisition of number concepts.

Before moving on, a final remark about the dispute between ANS- and OFS-based accounts of the acquisition of number concepts. I did not address mixed accounts here. It may be that children learn the first number words with the aid of the OFS, then they bootstrap the cardinality principle as a mechanical procedure, and only later is a mapping established onto the ANS, endowing words for numbers larger than four with meaning. Accounts of number learning that combine contributions from both the ANS and OFS have been put forward by Spelke (2011, p. 304-305), Pantsar (2014, 2015), and vanMarle et al. (2018).

4.6 Marks of numeral systems in the encoding of cardinal values

We have seen that, according to findings from developmental psychology, numerals used in the counting procedure are the main elements responsible for engendering in us the idea of exact cardinal values, which constitute the initial meanings we associate to numerals themselves. That is, numerals produce their own meanings.[8] In the ANS-based account, symbols for numbers have the power of sharpening ANS representations of numerosity, transforming them into proper cardinal values. In the OFS-based account, symbols for numbers play a double role: first, they promote the fixation of the OFS's implicit representations of numerosity in long-term memory, where they become cardinal values associated to numerals from 'one' to 'four'; second, they promote bootstrapping. In this section, we will see other experimental results that lend further support to the view that numerals engender their own meanings. These experimental results show that the particular symbolic system through which we learn numbers leaves permanent marks on the way we encode cardinal values.

Perhaps the most notable mark of symbolic systems on the encoding of number concepts is the so-called SNARC effect (SNARC stands for Spatial-Numerical Association of Response Codes). Dehaene, Bossini, and Giraux (1993) found the SNARC effect in a parity judgment task, where participants were presented with a digit and asked to press one of two buttons depending on whether the digit referred to an even or odd number. The buttons were placed side by side, and the assignment of odd and even to the right and left buttons was systematically varied, so that participants were tested both in situations left-even/right-odd and left-odd/right-even. The authors found that, in French participants, larger numbers elicited faster responses with the right-hand button, whereas smaller numbers elicited faster responses with the left-hand button (regardless of the numbers being even or odd). By contrast, amongst Iranian participants, who were raised in a right-to-left writing culture, the effect reversed. They concluded: "[t]he particular direction of the spatial-numerical association seems to be determined by the direction of writing" (Dehaene et al., 1993, p. 394).

[8] Following the practice in cognitive science, I am using the term 'meaning' to designate the mental content activated by a symbol.

The association between direction of writing and direction of the SNARC effect has been confirmed in several studies since then. Zebian (2005) tested literate Arabic speakers, who write from right to left, and illiterate Arabic speakers, who were not able to read and write but were able to recognize digits. Literate Arabic speakers showed a reverse SNARC effect, whereas no effect was observed in illiterates. Shaki, Fischer, and Petrusic (2009) showed that Palestinians, who read Arabic words and Eastern Arabic digits from right to left, display a reversed SNARC effect (small-right/large-left), whereas Israelis, who read Hebrew words from right to left but Arabic digits from left to right, display no reliable association between number and space. Li, Zhang, Zhang, Fanga, and Li (2017) tested Chinese speakers with different number notations they were familiar with (Arabic digits, simplified Chinese, and complex Chinese numerals) and concluded that the direction of the SNARC effect is notation-specific.

The dominant explanation of the SNARC effect holds that cardinal values are encoded along a "mental number line" whose orientation is determined by reading and writing habits (Dehaene et al., 1993). An alternative hypothesis holds that the spatial orientation is not encoded in the mental number line, but is rather constructed in working memory, during task execution (van Dijck & Fias, 2011). Either way, the existence of the SNARC effect shows that mathematically irrelevant features of numeral systems—broadly conceived, also encompassing the writing tradition in which they are embedded—interfere with the way we encode or process numerical information.

Other mathematically irrelevant features of the decimal place-value system of Arabic digits also influence the encoding of number concepts. This is illustrated by the *unit-decade compatibility effect*, which appears in digit magnitude comparisons. A two-digit number comparison is said to be *compatible* if both the decades and the units of the involved numerals bear the same less-than (or greater-than) relation to each other. For example, the comparison 42 vs. 57 is compatible because $4 < 5$ and $2 < 7$; by contrast, the comparison 62 vs. 46 is incompatible, because $6 > 4$ but $2 < 6$. Nuerk, Weger, and Willmes (2005) found that incompatible comparisons are slower and more error-prone than compatible ones. This means that, when comparing two numbers, we do not simply compare cardinal values encoded in a notationally irrelevant way. Nuerk and colleagues suggest that, under the influence of the decimal system through which we learn numbers, the mental number line is decomposed into different segments for tens and units.

Nuerk, Moeller, Klein, Willmes, and Fischer (2011) list 15 other effects observed in mental operations with numbers whose explanations might hinge on characteristics of the decimal place-value system. These effects range from the observation that mental computations with multiples of ten are easier than computations with other numbers, to less obvious effects such as the *decade crossing effect*, which appears in the *number bisection task*. In this task, participants are presented with a number triplet and are asked to evaluate whether the central number of the triplet bisects the interval defined by the outer numbers. For example, in the triplet 21 25 29, 25 bisects the interval between 21 and 29, whereas in the triplet 21 23 29, 23 does not. Mathematically, this problem can be solved in a uniform way to any given triplet (e.g, one can subtract the outer numbers from the central, and compare the results). However, it has been observed that, when the triplet crosses a decade boundary (e.g., 29 32 35 crosses the decade boundary 30), participants perform more poorly than when the triplet is within the same decade (e.g., 31 34 37). Again, this result indicates that features of the notational system, and not just the cardinal values of the numbers, are involved in solving the task.

The existence of effects such as these demonstrates that mathematically irrelevant features of the symbolic system we use interfere with the way cardinal numbers are encoded. Conversely, when the symbolic system lacks the required mathematically relevant features, number concepts are unlikely to arise. Mundurucu is a language spoken by an Amerindian group that lives in the Amazon forest in Brazil. Mundurucu is described as having numerals only for numbers from one to five. Above this limit, Mundurucu speakers use approximate quantifiers that could be translated as "some," "many," or "a small quantity." However, even their numerals (with the exception of the words for one and two) are not used consistently to refer to exact quantities.

> For instance, the word for 5, which can be translated as "one hand" or "a handful", was used for 5 but also 6, 7, 8, or 9 dots. Conversely, when five dots were presented, the word for 5 was uttered on only 28% of trials, whereas the words for 4 and "few" were each used on about 15% of trials (Pica, Lemer, Izard, & Dehaene, 2004, p. 500).

Pica and colleagues also report that Mundurucu speakers do not attribute a numeral to a collection based on the outcome of a counting procedure. Rather, "[o]ur measures confirm that they selected their verbal response on the basis of an apprehension of approximate number rather than on an exact count" (Pica et al., 2004, p. 500). Dehaene, Izard, Spelke, and Pica (2008) tested monolingual speakers of Mundurucu and native speakers of Mundurucu who also spoke Portuguese in a number line positioning task. Participants were presented with different kinds of numerical stimuli—clouds of dots, Mundurucu numerals, and Portuguese numerals—and then asked to indicate on a line segment, with one dot at the left end and ten dots at the right end, where that numerical stimulus should be placed. For both Mundurucu numerals and clouds of dots, participants placed stimuli in the line segment according to a logarithmically compressed scale, consistent with ANS representations of numerosities. Bilingual participants, though, placed stimuli conveyed in Portuguese number words according to a linear scale with fixed variability, consistent with the arithmetical number line, but they still placed stimuli conveyed in Mundurucu numerals according to a logarithmic scale. These results show that "neither linguistic competence per se, nor numerical vocabulary and verbal counting suffice to induce the log-to-linear shift" (Dehaene et al., 2008, p. 1219). The point is that the numeral system of Mundurucu is approximate and "does not emphasize measurement or invariance by addition and subtraction as defining features of number, contrary to Western numeral systems" (Dehaene et al., 2008, p. 1219). Mundurucu "numerals" seemingly refer to ANS representations of numerosities, rather than to exact quantities. The fact that even bilinguals who have number concepts (acquired through Portuguese numerals) understand Mundurucu "numerals" as referring to approximate quantities reinforces this conclusion. Approximate "numerals" cannot sharpen numerosity representations.

Taken together, the observations that mathematically irrelevant features of numeral systems leave marks on the mental encoding of number concepts and that the absence of a mathematically relevant feature—exactness—precludes the emergence of number concepts further support the view that number concepts are, to a significant extent, a product of numeral systems.

4.7 Counting as a cognitive tool

The results reviewed above show that the practice of counting is indispensable for the acquisition of concepts for numbers larger than three or four. The counting procedure makes use of a list of symbols for numbers but, as we have seen, numerals for numbers above three or four will become meaningful for those who are learning to count only *after* they have mastered the counting procedure. In section 2.4, I recruited Krämer's (2003) description of operative writing and de-semantification to explain how it is possible to master a symbolic system without knowing what its symbols mean. Now, we are in a position to better understand how this happens during the process of learning numbers.

Counting can be described as a symbolic system of operation similar to Krämer's systems of operative writing, with the difference that counting is oral. When counting, we do not write down symbols, but utter words as a means to fulfill a cognitive task. Similar to systems of operative writing, the counting procedure can be seen as consisting of two parts: a set of symbols and a set of rules that govern how the symbols should be manipulated (operational rules). The set of symbols consists of an ordered list of words. The operational rules are those codified in Gelman and Gallistel's counting principles, as presented in section 4.3. Recall that operational rules, as defined in section 2.4, specify the actions that must (or can) be performed so as to solve a problem. In the case of counting, the target problem is the determination of the cardinality of collections. The counting principles specify how this can be done.

Importantly, operational rules can be operated mechanically with de-semanticized symbols, without the agent needing to know either the intended meaning of the symbols or the purpose of the rules she is operating with. This is why children can master the counting principles before they have conceptual understanding of what they are doing. At earlier stages, they simply "sing-song" counting words while pointing to objects. As we have seen, children start mastering the procedural rules of counting (principles 1 to 4) when they barely understand what 'one' means. And, as Davidson et al. (2012) and several other studies point out, even the cardinality principle is first learned as a mechanical procedure.

The importance of counting for the acquisition of number concepts becomes more salient for values larger than three or four. In principle, the meaning of the first three or four number words can be acquired without the aid of the counting procedure (as noted by Carey et al. (2017, p. 244), quoted above). Perception of numerosities up to three or four is accurate. Thus, children "merely" need to learn to associate the words 'one,' 'two,' 'three,' and 'four' with their corresponding numerosities. In fact, as we have seen, subset-knowers correctly understand the cardinal values associated with the first three or four number words even though they have not yet mastered the counting procedure. Above four, however, perception of numerosities is vague. There are no clearly distinguishable numerosities with which the words 'five,' 'six,' 'seven,' etc., could be associated. This is where the counting procedure becomes indispensable.

What children need to carry on the formation of number concepts is a method to evaluate larger cardinalities as accurately as subitizing. Numerate adults use numbers for this. But children who are just starting to learn the first numbers do not know larger numbers yet. A method of determining cardinality that is available to agents who do not understand numbers is one-to-one correspondence. And counting with de-semanticized symbols is, in fact, a method of one-to-one correspondence.

If we view counting words as de-semanticized, self-contained words that do not refer to anything besides themselves, then the list of counting words becomes a model collection similar to any other collection of objects used for this purpose. Recall that subset-knowers, when asked how many objects there are in a collection they have just counted, do not answer with the last word they had used; instead, they recount the set (Wynn, 1990). This is tantamount to displaying the whole model collection in order to convey information about the size of the target collection, which is exactly what one needs to do if she is using one-to-one correspondence.

Counting becomes different from mere one-to-one correspondence when the cardinality principle is in place. In this case, the last word used in a counting episode gains a proper meaning; it refers to the size of the whole collection. And this size is exact, since the counting of collections with one more or one less item results in different last words. Over time, the recurring association of a word with the same cardinal size endows the previously meaningless word with meaning (how this happens is explained by the ANS- or the OFS-based accounts of the acquisition of number concepts).

We begin the journey of learning numbers equipped only with subitizing and estimation. Subitizing helps us learn the meaning of the first number words. To go on, though, we have to learn a mechanical procedure of one-to-one correspondence (counting with de-semanticized number words) that allows us to extend the accuracy of subitizing to larger collections (more on this in section 5.3.1). Without this intermediate step where we use counting as a mechanical, de-semanticized procedure, we could not obtain the idea of exact cardinal values above the subitizing range. In this sense, counting, the cognitive tool composed of a list of words and a set of operational rules, is constitutive of concepts for numbers above three or four. Now we can see that counting is constitutive of number concepts in ontogeny precisely because it can be de-semanticized. De-semanticized, the counting procedure can be learned by those who do not know numbers, giving them the opportunity to develop number concepts.

4.8 Conclusion

We have seen that the ontogenetic development of numerical cognition could not take place without the involvement of numerals and counting. In the ANS-based account of number acquisition, the acquisition of symbols for numbers has the effect of sharpening numerosity representations to the point of transforming them into exact cardinal values. In the OFS-based account of number acquisition, symbols for the first three or four numbers have the effect of promoting the storage of long-term representations of small numerosities, which become the meanings of the first number words. At a second stage, when the counting procedure is mastered, children generalize their knowledge about the first number words to build cardinal values for the subsequent number words.

Children do not discover numbers by themselves, through their personal experiences with the world or through the maturation of their quantical skills. Rather, they learn numbers from other people. Numeral systems and counting techniques are cultural creations. Learning numbers is a process of enculturation. Phrased in Vygotsky's terms, a child learns to count when an interpersonal process—a tutor explicitly using numerals, showing how to count and inviting the child to count together—is converted into an intrapersonal one—the child becoming able to count by herself. Innate quantical abilities may help children understand what is going on, but the crucial ingredients for numerical cognition—number words and the counting procedure—come from the interpersonal relationship with tutors. This means that number concepts are passed on through cultural transmission: those who already have number concepts transmit them to the next generation by teaching numerals and the associated techniques. But this leads to a natural question: from whom did the first generation of number-knowers learn numbers? The historical origins of number concepts are the topic of the next chapter.

Chapter 5

The historical origins of number concepts

CHILDREN create a new conceptual system when they acquire number concepts. Nativist accounts and terminological confusions aside, this is a consensus view in developmental psychology, as we saw in Chapter 4. It is also widely accepted that the creation of this new conceptual system relies on the availability of a symbolic system for numbers in children's environments. If children are not taught a symbolic system with the right features—words referring to exact discrete quantities—they will not develop number concepts. This poses a problem for the explanation of the historical origins of numeral systems. When no numeral system had been created yet, people did not have the opportunity to acquire number concepts. But, if people did not have number concepts, how could they ever create a symbolic system *for numbers*?

It seems that at some point in the past, there must have been someone who somehow managed to acquire number concepts in the absence of any previously available numeral system and then invented the very first symbolic system for numbers. But developing number concepts in the absence of a previously available numeral system is unlikely according to ontogenetic theories of number acquisition, and inventing a numeral system without previously available number knowledge seems equally unlikely. Pelland (2018a) calls this paradoxical situation "the origins problem."

I start this Chapter, in section 5.1, by presenting Pelland's formulation of the origins problem. Then, drawing on Dutilh Novaes's (2012) account of re-semantification, I advance a hypothesis about how the first numeral systems may have been created by anumeric peoples. The remaining sections of this chapter bring evidence from linguistic studies with anumeric and few-number cultures to provide evidence for the hypothesis proposed in section 5.1. In section 5.2, I give an account of the emergence of the words for one, two, and three in which people's need to refer to the numerosities they perceive through subitizing paves the way for the emergence of the first pairs of number words/number concepts. The etymology of the words for one to three used in a family of Amazonian languages, known as Nadahup or Maku, illustrates this point. In section 5.3, I discuss how people who only know numbers up to three or four could have invented the first one-to-one matching techniques to assess the cardinality of larger collections. Tallying systems used by contemporary few-number cultures illustrate the practical needs that may have motivated the creation of tallies. In section 5.4, I discuss the body-part tallying systems used by speakers of Nadahup languages which illustrate how the re-semantification of words used to describe tallying gestures gave rise to number words and number concepts in these cultures. In section 5.5, I explain how verbal counting systems with a low upper bound can grow towards infinity.

5.1 The origins problem

Since Chapter 2, I have been emphasizing that there is a gap between quantical and numerical cognition. As Núñez (2017, p. 420) puts it, "quantical phenomena, being the product of natural selection alone, cannot scale up to numerical phenomena (i.e., exact, relational, operational) across the symbolic reference gap." The *gap problem*, as Pelland (2020, p. 3798-9) describes it, is the question of how we "we bridge the gap between the rudimentary numerical [quantical] content produced by our evolutionarily ancient brains and the arithmetically-viable numerical content that comes to be associated with numeration systems like Indo-Arabic or Roman numerals." ln ontogeny, as we saw in the previous chapter, this gap is bridged by the acquisition of numerals and the counting procedure. This is what explains, at the individual level, what makes an individual equipped only with quantical skills, and therefore unable to clearly distinguish ten from eleven, become competent with numbers and therefore able to distinguish between any given cardinal sizes (if sufficient time for counting is provided). But numerals and the counting procedure are cultural creations and, as such, must have been created at some point in the past. Pelland's (2018a; 2018b) point is that *externalist* explanations of how the gap is bridged in ontogeny (i.e., explanations that rely on the availability of culturally-created symbols and techniques in the cultural environment where individuals are raised) do not explain how the gap was bridged in history. The *origins problem* is the historical side of the gap problem: "how … can we rely on external symbols for numbers in our explanation of the development of numerical content when the existence of such symbols in turn depends on the existence of number concepts?" (Pelland, 2018b, p. 185). For Pelland, the difficulty externalist accounts face in explaining the historical origins of numerical competence should count as counterevidence against the externalist explanation in ontogeny as well.

> if we are trying to explain the ontogeny of number concepts, our theory should apply to *everyone* capable of thinking about numbers. But since some people seem to have been able to think about numbers without external aids in the (distant) past, any account that depends on such support will not apply to every case of numerical cognition. At best, such externalist accounts could describe how numerical cognition emerges in a numeral-enriched environment. Even so, the fact that it is possible to develop some basic number concepts without external support seems to suggest that cases that do involve external support might somehow appeal to a more fundamental process, which the externalist framework is leaving out. … the ontogeny of number concepts in a world where symbols for numbers abound cannot be completely separated from past cases of numeral-free ontogeny since the former depends on the latter in important ways (Pelland, 2018b, p. 185-186).

Pelland is right to require that explanations of the ontogenetic development of numerical cognition be compatible with explanations of the historical emergence of number concepts. These stories are not independent of each other, since the pioneers of numerical cognition must have experienced an ontogenetic process at least similar in crucial aspects to the one today's children undergo. In face of this, Pelland suggest that the

> externalist approach needs to be replaced by one that focuses more on internal cognitive processes, since any appeal to external symbols for numbers must come *after* we have explained the emergence of numerical cognition *internally*, given that external symbols for numbers depend on the construction of internal representations with numerical content for their existence (Pelland, 2018b, p. 179).

The problem with Pelland's suggestion is that such an internal cognitive capacity to engender number concepts in the absence of external aids has never been detected.[1] On the contrary, it is a well-established fact that numerical cognition does not arise without the aid of external symbols and practices. To be in accordance with the available evidence, then, any explanation of the historical emergence of numerical cognition should be built around this fact. However, the possibility of giving a purely externalist historical account of the origins of numerical cognition is a non-starter if "external symbols for numbers depend on the construction of internal representations with numerical content for their existence" (Pelland, quoted above). How can we get out of this paradoxical situation?

The first thing we have to take into account is that symbols are not only a means of expression and denotation. If we think of symbols only as a means to express ideas that were previously engendered in someone's mind or to denote things that exist "out there," then it seems impossible that a symbol could come into existence before the emergence of the idea or thing it refers to. But, as we saw in section 2.4, symbols can also be used for operational purposes and, used in this way, symbols have the potential of giving rise to new contents that are inaccessible otherwise. Thus, it is at least in principle possible that symbols for numbers preceded number concepts.

Re-semantification (Dutilh Novaes, 2012) is one of the processes through which symbolic systems can give rise to new contents. Through re-semantification, a symbolic system originally dedicated to one conceptual domain can give rise, under certain conditions, to a new conceptual domain whose emergence was suggested by the operational use of the very same symbolic system. This opens up the possibility that the symbols that eventually gave rise to numerals were not symbols for numbers since the beginning. It may be that originally *non-numerical* symbols have been used for cognitive operations that ended up engendering number concepts. These initially non-numerical symbols might have come from other contexts, where they had other meanings. For the emergence of number concepts, it is sufficient that these symbols and the cognitive operations they facilitated produced in our ancestors the same kind of stimuli that triggers the development of number concepts in today's children. Once these originally non-numerical symbols had given rise to number concepts, they may have been co-opted to express the novel ideas they had helped bring up. In other words, numerals and number concepts may have appeared together, but the symbols that ended up becoming numerals likely preceded number concepts.

[1] Unless one assumes a nativist stance. For the nativist, there are no gap nor origins problems.

The operation of a mechanical procedure following the five counting principles is what triggers the development of concepts for numbers above three or four in today's children. We saw in Chapter 4 that children first learn to execute the counting procedure mechanically; only later does conceptual understanding of numbers come about. A cognitive task similar to counting, but one that does not require prior competence with numbers for its invention, might have served the same function for our ancestors. The natural candidates are tallying techniques based on the principle of one-to-one correspondence. It is easy to see that tallies are regulated by at least three of the counting principles: one-to-one correspondence, order irrelevance of pairing acts, and abstraction. The two remaining principles—stable order and cardinality—may have been suggested by a particular kind of tally, viz., body-part tallies, as we will see in section 5.4. The suggestion is that, by using purely mechanical tallying procedures regulated by the five counting principles, our ancestors may have ended up activating the cognitive processes that engender number concepts. Then the symbols they used for tallying were re-semanticized to refer to number concepts.

When it comes to the first three or four pairs of numerals/number concepts, the solution may be a bit simpler. We saw in Chapter 4 that the acquisition of the first three or four number concepts takes place before the counting procedure is fully mastered. This means that, in a hypothetical historical picture of the first emergence of number concepts, the concepts for one, two, and three may have emerged even before the invention of tallying techniques, although they cannot have emerged independently of symbols to refer to the corresponding numerosities. One possibility is that words that originally designated particular collections with the relevant numerosity ended up being co-opted to refer to the size of these collections, thus giving rise to the first number concepts.

In this way, the emergence of number concepts in history is made compatible with the emergence of number concepts in ontogeny without needing to postulate hitherto undetected internal faculties. In both cases, the gap is bridged by external symbolic systems and practices.

These are hypotheses about historical processes and, as such, are in need of empirical support. To provide the required evidence, however, is not a simple task, since the history of today's fully-fledged numeral systems is lost to the past. As Hurford (1987, p. 82) puts it "[t]he remoteness in time and space of the origins of numeral systems, together with the possible effects of cultural and linguistic mixing and borrowing, make the evolutionary picture hard, if not impossible, to discern by any method resembling direct observation." Where direct observation fails us, we need to recruit other methods. Linguistic studies of contemporary anumeric or few-number cultures, such as the Pirahã (D. Everett, 2005) and the speakers of Nadahup (aka Maku) languages (Epps, 2006, 2013), are a valuable source of data for this task. As we will see in the next sections, evidence gathered from these and other peoples and languages lends support to the hypotheses outlined above.

5.2 The first number words: one, two, three

Sometimes it is claimed that the faculty of subitizing could deliver the first number concepts without extra effort. However, evidence from developmental psychology suggests otherwise. As we saw in Chapter 4, children take about one year to acquire the meaning of the words for two and three after they have already acquired the meaning of the word for one, even though they have had the ability to subitize since their first hours of life. If the first number concepts were provided by the faculty of subitizing alone, it would be expected that children could attach number words and number concepts more easily, at least after the crucial realization that numbers words refer to quantities, which takes place when they grasp the meaning of 'one.' The OFS (and possibly the ANS too) plays a central role in the acquisition of concepts for the smallest numbers, but it does not deliver these concepts "for free."

This is further corroborated by studies with the anumeric Pirahã. The Pirahã are an indigenous people who live in the Amazon forest in Brazil. They speak the only known language (also called Pirahã) that lacks number words completely. The most numeral-like words they have are the words *hói*, *hoí*, and *baágiso*. At first sight, their use of these words would suggest a "one, two, many" system, where *hói* and *hoí* would denote exactly one and two, respectively. But closer inspection refutes this interpretation. Gordon (2004, p. 497) reports:

> Whereas the word for "two" [*hoí*] always denoted a larger quantity than the word for "one" [*hói*] (when used in the same context), the word for "one" was sometimes used to denote just a small quantity such as two or three or sometimes more.

In another study, Frank et al. (2008) confirmed that the words *hói*, *hoí*, and *baágiso* do not denote precise numbers by testing adult Pirahã speakers in a numeral elicitation task. They presented participants with varying quantities of objects successively, starting with one object and ending with ten, and vice-versa. In each trial, participants were asked "how many?". Whereas in the ascending order the candidate word for one, *hói*, was used only to refer to collections of one object, in the descending order participants started applying *hói* to collections of six items and continued doing so all the way down to collections of one item. In the ascending order, the candidate word for two, *hoí*, was applied to quantities ranging from two to ten items; in the descending order, it was applied to quantities ranging from four to ten items. "Because each of the three words was used for a dramatically different range of values in the ascending and the descending elicitations, these words are much more likely to be relative or comparative terms like 'few' or 'fewer' than absolute terms like 'one'" (Frank et al., 2008, p. 820). D. Everett (2005) proposes that the words *hói*, *hoí*, and *baágiso* translate into English as "small size or amount," "a couple or a few," and "many," respectively.

Other results reported in Gordon (2004) and Frank et al. (2008) show that the Pirahã can subitize as accurately as everyone else. Therefore, the conclusion must be that subitizing is not a sufficient condition for the emergence of words for the first cardinal values. But one could still argue that, even missing the words, they could have the concepts. However, this is highly unlikely, since the OFS by itself cannot provide such concepts, as we saw in section 3.5. Lacking symbols for numbers, they cannot undergo the conventional ontogenetic process of number acquisition. Perhaps non-verbal symbols, such as gestures, would do, but the Pirahã also lack gestures for precise small quantities (D. Everett, 2005). All this suggests that monolingual Pirahã speakers lack not only symbols for, but also the concepts of one, two, and three. Therefore, an explanation of how it was possible for people living in an anumeric culture to devise number concepts without previously having symbols for them must start with the concept of one.

The details of the creation of words for and concepts of the smallest numbers are lost to the remote past, but there are some clues that can help us tell a plausible story of their creation. A first insightful observation is that, in many languages, words for the smallest numbers inflect, i.e., agree in gender or case with other words in their syntactic environment. According to Hurford (1987, p. 114),

> [i]t is of the essence of a rote-learnt sequence of words that each word have a single form … So numeral words which originate in the recited rote-learnt sequence would be expected to be uninflected. The occurrence of variant inflected forms of words for 1, 2, 3 (and 4) suggests that these words originate in ways more closely integrated with their eventual use as modifiers of nouns indicating collections of things.

This is in line with the observation that subset-knowers learn the meaning of words for one, two, three, and possibly four before they have full command of the counting principles. Besides languages in which words for the smallest numbers inflect, there are also some languages in which the words used for small numbers in the counting sequence are completely different from the words used for the same numbers in other contexts. For example, in Kombai, a language spoken in New Guinea, the words for one and two are, respectively, *mofenadi* and *molumo* (or *lumo*), whereas in their body-based counting sequence the words for one and two are *raga* and *ragaragu*, respectively (De Vries, 1994). This reinforces Hurford's suggestion that the origin of words for the smallest numbers may not be related to the development of tallying and counting procedures.

Following Hurford's suggestion, Carey proposes that words for the smallest numbers may have been derived from grammatical number. Grammatical number differs from the word class of numerals. Grammatical number is a category of inflections that distinguish references to one item (singular) from references to more than one item (plural). Some languages also have inflections to distinguish references to collections of two items (dual) and three items (trial). In English, grammatical number is marked, e.g., by the use of the plural suffix -s and by the singular article 'a'/'an.' Carey proposes that

> historically, the initial meaning of "one" overlapped substantially that of the singular determiner "a," and … the initial meaning of "two" overlapped substantially that for dual markers in languages that have them, and the initial meaning of "three" overlapped substantially that for a trial marker (Carey, 2009, p. 323).

Carey's suggestion is cognitively viable, since the use of singular/dual/trial/plural markers does not require familiarity with numbers. To see why, consider how duals are used in Arabic. In this language, the dual form is obtained by the addition of a suffix to the singular form. Thus, in Arabic it is possible to say "I read two books" without using the Arabic number word for two by saying something such as "I read book.dual" (FCLangMedia, 2014). The concept of two—i.e., the idea of an abstract cardinal value, as described in section 4.1—is not required in order to understand the use of the dual suffix because it always appears attached to a noun, thus referring to a specific collection of objects. What is required from the user of a dual marker is only the ability to perceive collections of *a* and *b* distinct objects precisely, which is innately provided by subitizing. In fact, the same goes for the singular/plural distinction in English: because quantical cognition allows us to clearly distinguish a single object from collections of two or more, plural and singular nouns can be used independently of number concepts. Thus, cognitively speaking, grammatical number is non-numerical. However, users of grammatical number are halfway through developing number concepts, since they are already paying attention and referring to cardinalities.

Therefore, it is no surprise that Pirahã lacks not only numerals, but also grammatical number inflections completely. There is nothing like the singular/plural distinction or other manners of expressing numerical distinctions in Pirahã grammar. According to Everett, Pirahã sentences are intrinsically ambiguous with regard to number. For example, the sentence *hiaitíihí hi kaoáíbogi baiaagá* translates into English as "The Pirahã are afraid of evil spirits," or "A Pirahã is afraid of an evil spirit," or "The Pirahã are afraid of an evil spirit," or "A Pirahã is afraid of evil spirits" (D. Everett, 2005, p. 623).

The double lack of numerals and grammatical number in Pirahã can be seen as corroborating Carey's claim that grammatical number inflections can give rise to numerals/number concepts. Nevertheless, this cannot be the only way of creating words/concepts for the smallest numbers, since there are languages, such as Chinese, that have numerals but do not have the singular/plural distinction.

Another process of creation for concepts/words for the smallest numbers is suggested by Epps's (2006) studies of the development of numeral systems in Nadahup languages. The Nadahup family includes four documented languages—Nadëb, Dâw, Hup, and Yuhup—whose numeral systems' upper limits range from three to 20. Interestingly, Nadahup numerals still preserve transparent etymologies, "a cross-linguistically unusual feature suggestive of their relatively recent development" (Epps, 2006, p. 259). Nadahup numerals for the smallest numbers are not derived from quantifiers or grammatical number markers, but from words with other, non-numerical, meanings. In Hup, the number word for one—*ˀayǔp*—seems to be derived from the demonstrative pronoun '*yúp*' (meaning "that"), which is used for abstract, absent or intangible entities. In Yuhup, the number word for one—*cā̂h*—also means "other," and a variant of it—*cãhyā̂pã*—means "other individual." The words for two and three reveal even more interesting etymologies. In Hup, one of the variants of the word for two—*kəwəg-ˀap*—literally means "eye quantity," and one of the variants of the word for three—*mɔ́t-wɨg-ˀap*—literally means "rubber tree seed quantity." Eyes are a universal paradigmatic case of a collection that always comes in pairs, whereas rubber tree seeds are familiar triplets in Hup speakers' environment.

> The rubber tree (*hevea* sp.) has a large, distinctive, three-lobed seed or nut (in Hup, *mɔ́t-wɨg*) which is culturally highly salient, being used among the Hupd'əh and other peoples of the region to make a popular children's toy, and associated with an edible fruit (Epps, 2006, p. 264).

The Hupd'əh are the people who speak Hup. A rubber tree seed is depicted in Figure 5.1. But how could rubber tree seeds and pairs of eyes give rise to number concepts? Decock (2008, p. 464) suggests that anumeric societies could use immutable collections of permanent objects—which he calls "canonical collections"—as standards against which other collections can be compared by one-to-one correspondence. This seems to be exactly what Hup speakers have done. The distinct lobes *a*, *b*, and *c* of a rubber tree seed can be put in one-to-one correspondence with the distinct objects *d*, *e*, and *f* of another collection, and then the phrase "rubber tree seed quantity" can be used to refer to the cardinality of the latter. It must be noted that this one-to-one correspondence operation does not need to be actively conducted by the agent. The OFS can do this automatically. The only thing agents needed to actively do was to select the rubber tree seed as a canonical collection.

Figure 5.1: A rubber tree fruit (in Hup, *mɔ́t-wɨg*) with its three distinctive lobes. The Hup word for three is *mɔ́t-wɨg-ˀap*, where the suffix *-ˀap* means quantity, showing that Hup speakers clearly distinguish the fruit from the cardinal value represented by it. (Photo from ProjetoPETRA (2017).)

In doing so, over time, the association between the phrase "rubber tree seed quantity" and collections of three, according to the OFS-based account of number acquisition, would cause enriched parallel individuation to store a long term representation of a collection of the three lobes of rubber tree seeds, which thus becomes the meaning of the phrase/word "rubber-tree-seed-quantity." Later, this representation becomes fully abstract, and then the concept of three emerges. In the ANS-based account of number acquisition, the ANS representation of numerosities consisting of approximately three items would be sharpened by the association of the phrase "rubber tree seed quantity" with collections of three, giving rise to the concept of three.

The emergence of new numerical meanings left detectable marks on the corresponding words. The first thing to notice is that the Hup words for two and three are not simply "eyes" and "rubber tree seed;" they have the suffix *-ˀap*, which means quantity. This shows that Hup speakers clearly distinguish pairs of eyes and rubber tree seeds from the quantities they represent. Epps points out that a similar process of abstraction also occurs in the formation of words for colors.

> Especially for cultures where color reference carries a low functional load, color terms may come about via a gradual delinking and generalization of the abstract notion of color from specific objects—but this may be a slow process (Lyons, 1999; Kay and Maffi, 1999; Levinson, 2001). In the Papuan language Yélî Dnye, for example, the two dialectal variants of terms for 'red' correspond to the two different words for a species of red parrot that exist in these dialects; as Levinson points out, the co-existence of these terms suggests "*that the reference to the bird is still salient, that these are partially live rather than fully dead metaphors*" (Levinson, 2001: 18) (Epps, 2013, p. 344).

Again, we see numerosities and colors involved in similar phenomena. In Hup, although the references to eyes and rubber tree seeds are still visible in some variants of the words for two and three, other variants are already showing signs of detachment from their roots, becoming "dead metaphors." For example, *kəwəg-ˀap* (literally "eye-quantity") has the shortened form *kaˀap*. Phonological reduction, the process of lexicalization that is at play here, is the same process that in English produced the word 'goodbye' from the sentence 'God be with you' ("Goodbye", n.d.). In modern English, the concept of goodbye is completely detached from its roots, as in Hup the concept of two, expressed by the word *kaˀap*, is becoming detached from "eye-quantity."

The use of canonical collections to designate the numbers two and three is also present in the other Nadahup languages (except for Nadëb) and in other Amazonian languages. For example, in Xerénte, a language of the Je family, spoken thousands of kilometers away from the region where Nadahup languages are spoken, the etymology of the word for two—*ponkwanẽ*—traces back to a phrase that translates into English as "deer footprint," and the etymology of the word for three—*mrẽpranẽ*— traces back to a phrase for "rhea bird footprint" (Melo, 2007, p. 102). Deer footprints have two distinctive toes, and rhea birds' footprints have three toes.

The strategy of using canonical collections is less evocative for one, since each clearly distinguishable individual object may be as salient as every other when it comes to quantity. In Nadahup languages, the etymology of the words for one refers to "that" or "other individual," as mentioned above. Epps, Bowern, Hansen, Hill, and Zentz (2012, p. 66) report that in many Australian languages, the word for one also means "alone" or "together" (no parts); in the Amazonian language Xerénte, the word for one—*smĩsi*—also means "alone" (Melo, 2007, p. 102). Although the meaning of all of these words involves some reference to the idea of unit, notice that such ideas do not presuppose, and are different from, the concept of one. Recall that the minimum content of a number concept is a cardinal value—in the case of one, the idea of the cardinal size of a singleton consisting of a generic individual. We do not need this concept to be able to distinguish individuals we refer to by "that" or qualify as "together" or "alone." Only when one of these words is selected to refer to the perceived numerosity of singletons does the process of number concept formation take place and the concept of one arise.

Subitizing makes small quantities salient in the environment and this may be seen as a natural invitation to refer to these quantities in everyday situations. However, the salience of small quantities alone is not sufficient to elicit the creation of the first pairs of number concepts/words, as the existence of the anumeric Pirahã demonstrates. In fact, we do not feel compelled to refer to everything we are able to perceive. There must be other environmental factors that also drive the creation of number concepts, including factors from the social environment.

The emergence of number concepts/words is often viewed as being associated with the need to quantify scarce resources. A tension between multiple ends and limited supplies is believed to be what encourages people to engage in acts of quantification (Harper, 2008, p. 106). In a cross-cultural study involving data from more than 200 societies, Divale (1999) established a correlation between susceptibility to periodical starvation—which requires storing food for the future—and the upper limit of numeral systems. Generally speaking, the more food a society has to store, the higher the limit of its numeral system. Thus, a society that does not use numbers at all must be a society with no need to store food at all.

The Pirahã habits seem to be in total agreement with Divale's findings. D. Everett (2009, p. 76) reports that "[t]he Pirahã consume everything as soon as it is hunted or gathered." In fact, most of the features of the Pirahã's small-scale society—only 90 people in 1970 and 592 in 2014 (Gonçalves, 2018)—and simple material culture are compatible with not needing numbers. However, there is an additional difficulty in the case of the Pirahã. They have been in contact with Brazilian mainstream culture for more than 200 years, and they engage in barter frequently with Portuguese speaking traders who regularly come to their villages. It is well-documented that many cultures have acquired numeral systems through contact with other groups with which they maintain trade relations. This phenomenon is also common among Amazonian cultures. Epps (2006) points out that increased trade relations among Hupd'əh, Yuhup, and Tukanoan people has contributed to the diffusion of numerical strategies among them. More recently, these languages have also begun borrowing Portuguese numerals to refer to numbers above the limit of their original systems. Despite the long time the Pirahã have been in trade contact with numerate cultures, they have never borrowed numerals. According to D. Everett (2005), this points to a more fundamental characteristic of Pirahã culture: the Pirahã value referring only to immediate experience over abstract, unwitnessed topics. The Pirahã's limited interest in planning for the future, due to their attachment to immediate experience, might make them less interested in developing or adopting techniques to keep track of numerosities and, to the extent that the concept of number involves abstract generalizations that go beyond immediate experience, it might be experienced as a unfamiliar way of thinking in their culture.[2] If Everett is right, then, not only the material environment and the practical needs linked to it, but also other cultural traits can promote or preclude the emergence of number concepts.

Under the pressures of increasing practical or cultural needs, cultures that have broken the barrier of words/concepts for the first three or four numbers will be motivated to expand their list of numerals. However, this cannot be done by relying on the same strategies. Grammatical number is not apt for making precise distinctions above the subitizing range, since it is bounded by the OFS's limits. For the same reason, the selection of canonical collections of four or more items is not viable without the development of an active one-to-one correspondence procedure. The creation of tallying techniques is the answer.

5.3 Tallies: active one-to-one correspondence

In the Nadahup family, as we will see in section 5.4, words for numbers above three are derived from words recited during the operation of body-part tallies. Before addressing the emergence of words for numbers above three, though, we have to consider how tallying techniques in general, including non-verbal ones, might have emerged. A tally is a means of keeping track of discrete quantities by one-to-one correspondences without necessarily using number words or numbers. Ifrah (2000, p. 11) gives a graphic example of how people living in few-number or anumeric cultures can keep track of large quantities by tallying.

> Imagine a shepherd in charge of a flock of sheep which is brought back to shelter every night in a cave. There are fifty-five sheep in this flock. But the shepherd doesn't know that he has fifty-five of them since he does not know the number "55": all he knows is that he has "many sheep". Even so, he wants to be sure that all his sheep are back in

[2]The link that Everett establishes between the Pirahã's attachment to immediate experience and their lack of numbers is contested by Nevins, Pesetsky, and Rodrigues (2009). Their point is that there is nothing in the experience of seeing two canoes in the river and reporting "two canoes" that would violate an attachment to immediate experience. Although this may be the case, the absence of number words in Pirahã still demands an explanation that goes beyond the material/economical features of the Pirahã's lives, since they have been engaged in commerce for a long time and, even so, never developed nor adopted a numeral system.

> the cave each night. So he has an idea—the idea of a concrete device which prehistoric humanity used for many millennia. He sits at the mouth of his cave and lets the animals in one by one. He takes a flint and an old bone, and cuts a notch in the bone for every sheep that goes in. So, without realising the mathematical meaning of it, he has made exactly fifty-five incisions on the bone by the time the last animal is inside the cave. Henceforth the shepherd can check whether any sheep in his flock are missing. Every time he comes back from grazing, he lets the sheep into the cave one by one, and moves his finger over one indentation in the tally stick for each one. If there are any marks left on the bone after the last sheep is in the cave, that means he has lost some sheep. If not, all is in order. And if meanwhile a new lamb comes along, all he has to do is to make another notch in the tally bone.

Tallying techniques, implemented with materials such as bones, sticks, knotted cords, pebbles, fingers, and toes, have been used by many cultures across the world. Virtually anything that lends itself to be seen as a collection of discrete, easily individualizable objects or marks can be used as a tally. Tallies may have been one of the earliest non-verbal and non-numerical cognitive tools. As aptly illustrated by Ifrah, to operate a tally, one has only to master the mechanical procedure of establishing one-to-one correspondence. In his example, the model collection consists of the notches cut in the bone, and the target collection is the flock of sheep. To tally notches with sheep, the shepherd has to master the rule that governs one-to-one correspondence, namely: each item of the target has to be mapped onto a single item of the model, and for any *a* and *b* distinct items of the target, they must be mapped onto distinct items of the model. This means that he cannot assign a sheep to two notches, nor assign two sheep to the same notch. He also has to understand why this rule must be followed, otherwise he cannot appreciate its outcome. He has to be aware that each notch represents a single sheep, and that the collection of all notches represents the size of his flock.

As number concepts are not required for tallying, it is natural to suppose that anumeric people might be able to operate a tally. However, as tallying demands operational and conceptual knowledge of a procedure, and its execution requires a certain level of attention and control, we need to investigate whether humans are able to engage in tallying practices spontaneously or if previous training is required. The Pirahã are again a case in point. Frank et al. (2008) claim that, in spite of the Pirahã's anumeracy, they are able to establish one-to-one correspondences successfully if required to do so. Others, however, have not replicated their results. Let me briefly review the main points of contention in this issue.

Gordon (2004) tested some Pirahã on various one-to-one matching tasks. In the linear matching task, a number of AA batteries were placed in a line, and he asked participants to produce another row of batteries exactly below the presented one, with the same number of batteries. The orthogonal matching task was similar, but participants were asked to produce the other row of batteries in an orthogonal position, forming an angle of 90º with the presented one. In both tasks, participants performed accurately with up to two or three items, but beyond this performance deteriorated. Even though the orthogonal task proved to be more difficult for the tested Pirahã, performance on the linear task was also poor. Gordon's overall conclusion was that "[p]erformance with quantities greater than three was remarkably poor, but showed a constant coefficient of variation, which is suggestive of an analog estimation process" (Gordon, 2004, p. 496). That is, participants seemingly relied on their quantical abilities, rather than anything like a one-to-one correspondence procedure.

Frank et al. (2008) tested some Pirahã on the same one-to-one matching tasks, but found a different result in the linear task. In their experiment, participants performed almost at ceiling level when asked to produce another row of items exactly below the presented one, suggesting that they indeed applied a one-to-one correspondence procedure. Frank and colleagues claimed that Gordon's different result might be due to the AA batteries moving around inadvertently, which may have distracted the participants. To avoid distraction, Frank and colleagues used spools of thread placed vertically and uninflated rubber balloons. With these materials, the tested Pirahã performed much better on the linear task. However, they still performed badly on the orthogonal task. To explain participants' different performances on the linear and the orthogonal tasks, Frank and colleagues hypothesized that the orthogonal task demands more memory to transfer information across space. Lacking number words to encode the required information precisely, they had to rely on their ANS alone, and performed less accurately.

Frank and colleagues' results were contested in a follow-up study conducted by C. Everett and Madora (2012). They tested some Pirahã who lived in another village (further away from the one where Frank and colleagues had selected their participants) using the same procedures and materials, and obtained results consistent with those found by Gordon (2004). In their tests, participants failed on both linear and orthogonal matching tasks for collections with more than three items. C. Everett and Madora provide a plausible explanation for the disparate results. Besides the fact that the village where Frank and colleagues applied their tests (called Xagiopai) is more exposed to mainstream Brazilian culture—they report that there used to be a government-run clinic in the village—, they also reveal that

> in the 2 years preceding the field experiments of Frank et al. (2008), K.M. [K. Madora] spent months in the village. During this period, she sought to teach basic arithmetic to the people at their request. K.M. also used one-to-one matching tasks in her arithmetic sessions with the Pirahã at the Xagiopai village. … All of the adults at Xagiopai participated in such tasks numerous times in 2006, in the months leading up to the field research on which the findings in Frank et al. were based. (K.M.'s presence in the village during this period is documented via flight logs.) We are able to catalog the names of the tribe members in question and have corroborated their participation in the research conducted for Frank et al. The Pirahãs' heightened performance on the one-to-one matching task in that study can plausibly be explained because of their exposure to the task in question (C. Everett & Madora, 2012, p. 137-138).

If C. Everett and Madora (2012) are right, participants in Frank's et al. (2008) experiments succeeded at the line matching task because they had been previously trained. Although Frank was not impressed by C. Everett and Madora's arguments (see Frank (2012, p. 225)), another study conducted by his group can be interpreted (against their own interpretation) as confirming the hypothesis that success on one-to-one matching does require training in a mechanical, non-linguistic procedure.

Frank, Fedorenko, Lai, Saxe, and Gibson (2012) tested numerate US citizens recruited among the MIT community on the same tasks Pirahã participants were tested on in Frank et al. (2008). To prevent participants from counting and relying on number words to encode and memorize information, they required participants to repeat the words of a radio news broadcast aloud while performing the tasks. They hypothesized that, without being able to count, English speakers' performance should mirror Pirahã speakers' performance.

On the line matching task, English speakers performed as well as the Pirahã did in Frank et al. (2008). However, in the orthogonal matching task, where the Pirahã had failed miserably, English speakers performed with almost perfect accuracy. Frank et al. (2012) explain English speakers' superior performance by the fact that, in this task, they did not rely only on estimation (as the Pirahã did) but also on "ad-hoc non-linguistic strategies" (Frank et al., 2012, p. 77). They say that, during debriefing, participants reported "trying to co-register objects one by one across sets in the orthogonal match task" (Frank et al., 2012, p. 82). This clearly shows that English speaking participants used a non-linguistic tool—one-to-one correspondence—in this task which the Pirahã did not know how to use in a similar situation. Altogether, the results from Frank et al. (2008) and Frank et al. (2012) make the curious suggestion that success at linear matching does not require training, whereas success at orthogonal matching does.

In face of this, C. Everett and Madora's explanation of the conflicting results in the linear task becomes more compelling. It is more natural to suppose that, rather than revealing a natural ability to establish linear, but not orthogonal, one-to-one matchings, Pirahã speakers' success on the linear task in Frank et al. (2008) was due to training they had previously received. It is likely that Madora used parallel rows of objects to illustrate one-to-one mappings, rather than orthogonal ones. The limited experience of the Pirahã with the recently learned technique prevented them from generalizing the procedure to other spatial arrangements.[3]

Even if more studies are needed to settle this debate, the currently available results do not allow us to assume that the ability to perform one-to-one correspondence arises without training. In historical terms, this means that one-to-one correspondence techniques had to be invented. Therefore, we cannot avoid the challenge of explaining how techniques of one-to-one correspondence could have been created from scratch.

[3]Although I focused this discussion on the linear and orthogonal matching tasks only, the experiments cited involve a number of other tasks. One of the questions underlying the debate among Frank and colleagues and C. Everett and Madora is whether their lack of number words would prevent the Pirahã from recognizing exact quantities above the subitizing limit. What they do not seem to take into account is that one-to-one matching tasks where both target and model collections are visible throughout the process are not the most appropriate method to investigate this question, given that these tasks can be solved mechanically and non-verbally. To my mind, the success or failure of the Pirahã in these tasks is irrelevant for the debate about the importance of number words for the recognition of larger quantities. The "nuts-in-a-can" task, in which Gordon (2004) and Frank et al. (2008) also tested Pirahã speakers, is much more effective for this purpose. In this task, the experimenter drops nuts one by one into a can in view of the participant. Then, the experimenter withdraws the nuts again one by one, asking the participant after each withdrawal whether any nuts remain in the can. The accuracy of Pirahã participants on this task decreased as the number of nuts increased, displaying a pattern consistent with the use of quantical abilities only. The English speaking participants in Frank et al. (2012), tested on this same task under the condition that prevented them from counting, performed quite similarly to the Pirahã. These results conclusively show that, lacking both number words to encode numerical information in memory *and an external, mechanical and non-verbal one-to-one correspondence strategy*, exact quantities cannot be recognized. If English speaking participants in Frank et al. (2012) were provided with pencil and paper, they would probably have performed better by using tally marks.

5.3.1 The invention of tallying systems

Tallies, such as the notched bone in Ifrah's example quoted above, are genuine cognitive technologies. Not only do tallies such as that work as a memory aid, by recording the tallied quantity in an enduring material, but they also expand our ability to evaluate cardinal sizes precisely. The mechanical execution of a series of repetitive simple steps following the principle of one-to-one correspondence enables us to extend the precision of subitizing to cardinal values above three or four. As suggested above, tallying systems that originally did not involve numerical concepts may have been the precursors to counting. But how could anumeric people, or people who only know numbers from one to three or four, have invented this cognitive technology?

Anthropologists who study processes of innovation in human cultures and biologists who study non-human animals' capacity to innovate concur that inventions usually result from the recombination of preexisting elements motivated by necessity (Laland, 2017; O'Brien & Shennan, 2010; Reader, Morand-Ferron, & Flynn, 2016). "Necessity is the mother of innovation" is an often cited motto in the area, although there are other factors, such as opportunity and luck, which can prompt innovation as well. The invention of tallying techniques, however, does not seem to be an exception to the rule; it was probably motivated by necessity.

The first thing to notice is that quantical abilities themselves can pose a problem for which the search for solutions may lead to the invention of tallies. Some have suggested that the exact perception of numerosities provided by the OFS can be felt as conflicting with the fuzzy perception of numerosities provided by the ANS. Thus, "[a]s we apply these different systems to the same objects, events and scenes, we appear to be driven to reconcile the representations that they yield" (Feigenson et al., 2004, p. 313). Barner (2017, p. 554) elaborates on this point:

> perception provides humans with an explanatory problem that the creation of symbolic number systems is meant to solve. This problem, confronted by humans from the beginning of our shared cultural history, can be expressed as follows: whereas our perception of quantity is noisy and subject to error, our perception of individual things is not. Consequently, despite our noisy representation of number, we have a strong intuition that collections in the world are made up of distinct individuals, such that they must contain determinate numbers of things that are subject to precise measurement. … Counting systems, I propose, were constructed by our ancestors to resolve this explanatory gap—to measure and keep track of the precise quantities that we knew to exist in the world, but otherwise are unable to precisely quantify.

Although the disparity between the exactness of subitizing and the fuzziness of estimation may, in itself, be seen as a problem to be solved, it is unlikely that, without other practical needs, our ancestors would have bothered to solve this problem. As the existence of an anumeric culture shows, this internal "conflict" is not sufficient to prompt the invention of counting. The natural attitude towards large collections is estimation, and estimation usually suffices, unless there is a real need for exactness. Once such a necessity appears, then people probably will notice the internal "conflict" and try to extend the exactness of subitizing to the range of estimation.

There are a number of activities in small-scale societies which may ask for exactness. One of the most obvious is the tracking of time, and many small-scale societies have developed tallying systems specially for this purpose. For example, the Korowai from New Guinea use a simple tallying device, called a *saündal*, to keep track of the passing days. De Vries (1994, p. 562) describes it.

> It consists of the rib of a leaf of the sago palm tree, into which a number of pegs or bits of wood have been inserted. When somebody invites someone else to a feast, for example, he will hand over [a] *saündal* together with the invitation; the person invited will take one peg out of the *saündal* every day, and when he has reached the last peg, which is twice as long as the others, the day of the feast will have arrived.

The Korowai are far from being anumeric (they use a body-part counting system whose upper limit is 38), but it is easy to see that anumeric people could operate devices such as the *saündal*. To operate it, it is sufficient to understand that to each peg corresponds a day, and to be able to take one peg out every day. Both abilities are provided by the OFS. Recall that the OFS allows infants and non-human animals to judge whether two small collections are the same or different with respect to their cardinal sizes. Using this ability, an anumeric operator of a *saündal* can understand that each peg, taken in isolation, corresponds to one day, i.e., that they are "the same" in quantical terms. The OFS also enables the anumeric operator to take out exactly one peg per day, instead of two or three. When there are no remaining pegs, which the operator can also perceive through her OFS, she knows that the feast day has come.

In fact, subitizing seems to be the quantical ability that makes the establishment of one-to-one correspondences possible. The aforementioned experiments with the Pirahã show that they relied on subitizing when collections were small. In the linear matching task, they performed accurately when the row of objects consisted of up to three elements. In the orthogonal task, they succeeded with up to two items (Gordon, 2004). Their failure with numbers above the subitizing limit indicates that they did not decompose the task into smaller tasks manageable with subitizing. For example, if they had thought of a row of six batteries as a shorter row of three juxtaposed with another row of three, they could have succeeded. More generally, any collection of objects can be decomposed into singletons which can be easily dealt with using subitizing. Instead of proceeding in this way, though, the Pirahã estimated the numerosity of the whole line of items and then tried to deploy a set with the same quantity by estimating it again. What the Pirahã did not realize is that they could have shifted their attentional focus from the whole collection to its parts. The ability to shift the attentional focus in this way does not seem to require training. However, that this can be done in order to successfully complete the task seemingly requires an insight.

Tallying techniques may have been invented when people realized that any two collections, no matter how big they are, can be compared if decomposed into singletons, which can be successively paired with each other, relying on successive subitizations for this. If inventions come from the recombination of previously available elements motivated by necessity, then anumeric people, or people who know only numbers from one to three or four, are in the position to invent tallies once necessity arises, given that they already have the elements to recombine: the ability to shift the attentional focus from a collection to its elements, subitizing, and collections of discrete objects that can be used as model collections to measure the cardinal size of other collections. There is no reason to doubt that human creativity is capable of assembling these elements to develop tallying techniques.

There is evidence that cultures with a very limited numeral system use tallying techniques to expand their numeral systems. For example, the people from Kiwai Island, in Papua New Guinea, speak a dialect of Southern Kiwai that has a base-two numeral system, in which the word for one is *na'u*, and the word for two is *netowa*. Words for larger numbers are compounds of these two by addition: three is *netowa na'u bi* (two one), four is *netowa netowa* (two two), five is *netowa netowa na'u* (two two one), and so on (Owens, Lean, Paraide, & Muke, 2018, p. 44). Although in principle this system does not have an upper limit, it is easy to see that the expression of larger numbers becomes increasingly cumbersome. For this reason, Kiwai islanders prefer to use tally sticks when dealing with larger quantities. According to Smith, as cited in Owens et al. (2018, p. 138),

> the most common method of keeping count of large numbers in traditional Kiwai society was the use of tallies. Tally sticks each representing an object could be kept in a bundle or tied to a string. Tallies were used to represent the number of heads captured by a man in battle, or the pigs killed in the bush, or any other significant number.

Smith also reports that the Kiwai used this tallying system in ceremonial feasts in order to keep track of the number of gifts they received, so that they could repay slightly more during the return feast, and also to determine the day when the return feast was to be held, by using a method quite similar to the Korowai's *saündal*:

> [at the end of the feast] the donor and recipient groups would part, each having a bundle with the same number of sticks. Each day, both groups would discard a stick until the bundle was exhausted and the pre-arranged day for the [return] feast had thus been reached (Owens et al., 2018, p. 138).

Simpler one-to-one correspondence procedures, without involving any device, can also be invented in the context of exchanges. In many small-scale societies, practices of gift giving involve the obligation to reciprocate (Mauss, 1990). To give a concrete example, among the Juruna, an indigenous people from the Brazilian Amazon forest, gifts such as arrows must be reciprocated in the same number as they were received (Ferreira, 2002, p. 57). Societies with a very limited numeral system and a social rule to reciprocate gifts in the same number are likely to find a solution in one-to-one matching. Another social practice present in some small-scale societies that can lead to the development of one-to-one matching is the requirement for an equal distribution of goods among its members. In the absence of arithmetical abilities, successive rounds of distribution where one item is given to each member at a time seem to be a natural solution. In fact, practices like this can be found among the Xérente, another indigenous group from the Brazilian Amazon forest. Melo reports:

> [w]hen the festival takes place in the village, family heads naturally form a circle around the food, which is delivered apparently in accordance with a principle of quantity/person ratio; for example, if a person receives four packages of coffee, all family heads should also receive four packages (Melo, 2007, p. 107, my translation).

Traditionally, the upper limit of the Xérente's numeral system is four. Today, they also use Portuguese numerals, but they still keep this traditional practice of the distribution of goods.

These simple techniques will be invented only if people feel the need to distribute things equally, or to reciprocate the exact number of gifts, or to keep track of passing days, or have other reasons to be interested in determining the exact cardinal size of collections with more than three or four items. But, once tallies are in use, people who master a tallying procedure are halfway through acquiring number concepts. Tallying methods can be seen as precursors to counting because they are cognitive tools governed by three out of the five rules that govern the cognitive tool of counting: the principle of one-to-one correspondence (obviously), the principle of order irrelevance of objects, and the principle of abstraction. In its tallying version, the principle of order irrelevance can be rephrased as stating that the order in which items of the target collection are paired with items of the model collection is irrelevant, i.e., it may change across different tallying events. In Ifrah's example quoted at the beginning of this section, a shepherd who tried to always assign the same sheep to the same notch in his tally bone would find himself in pointless trouble. The principle of abstraction, in turn, regulates tallies as long as people realize that collections of different sorts of objects can be tallied, a practice that can be found in small-scale societies as exemplified above.

But two of the counting principles may still be missing in tallies. One of them is the cardinality principle, which in its tallying version would say that the model collection obtained at the end of a tallying event should be seen as representing the cardinal size of the target collection. In the given examples, though, the model collections are not produced to determine nor to represent the cardinality of the target collections, but only to fulfill a practical need. When using a *saündal*, for example, the agent may not be especially interested in how many days have to go until the feast takes place, but only in getting there on the right day. The *saündal* is used to perform an operation, rather than represent. Although its operator could think of the collection of remaining pegs as representing a number, she does not need to do so in order to succeed in attending the feast on the right day. The other principle missing is the stable order of counting words. In its tallying version, this principle would say that the items of the model collection should be used in a stable order across different tallying events. In tallies such as those in the above examples, this would be not only counterproductive, but also very hard to do, given that the items (e.g., pegs) of the model collections are hardly distinguishable from each other.

Lacking these two principles, tallying systems such as those exemplified above are not likely to give rise to proper counting and number words/concepts.[4] As Hurford (1987, p. 79) points out, this is because "sticks, or buttons, or whatever, have no names, and therefore th[ese] system[s] provid[e] no names for the numbers themselves." Without symbols associ-

[4] Overmann, Wynn, and Coolidge (2011) and Malafouris (2010, 2013) claim that the sole use of material artifacts—such as tally sticks, clay tokens, etc.—could give rise to number concepts. However, the lack of two counting principles in tallying systems that rely exclusively on hardly distinguishable material artifacts makes this possibility unlikely, as I argued above. Model collections with distinguishable items, such as those employed in the system of clay tokens in Sumer (Schmandt-Besserat, 1992) could overcome this difficulty, but Malafouris's explanation of the emergence of number concepts by means of the Sumerian system of clay tokens is probably anachronic. Given the complexity of Sumerian society by the time the system of clay tokens was being developed, it is highly unlikely that they had not yet developed number words for larger numbers (Overmann (2016) calls attention to this point). The development of the system of clay tokens in Sumer is better seen as a response to the need to record numerical quantities in permanent media, rather than to the need to count, which could have been solved earlier with words or simpler techniques.

ated with quantities on a regular basis, ideas of cardinal values are unlikely to be created in the mind. What is needed for the emergence of number words and cardinal values is a tallying technique based on named objects, which could be paired with the items of the target collection always in the same order, and whose names could be recited along with pairing acts. Body-part tallying systems display these features.

> Typically, the fingers and other bodyparts involved [in tallying] have verbal names. While pointing at a bodypart sign for a number, the verbal name of that bodypart can be uttered (Hurford, 1987, p. 80).
>
> With this method [body-part tallies] … there are still no verbal names for numbers, but the various bodyparts, fingers, and so on could themselves be taken as signs for the numbers. And there is, at this stage, a conventional ordering of these bodypart signs (Hurford, 1987, p. 80).

In the next section we will see the relationship between body-part tallying systems and the emergence of words for numbers larger than three or four.

5.4 Four and beyond: tallies with body parts

Fingers, toes, and other body parts seem to have played a central role in the emergence of number words. Tallying systems based on body parts, where the name of each part is uttered along with a gesture pointing to it, have been documented in many cultures across the world. Especially the various body-part tallying techniques found in New Guinea are often cited in accounts of the history of numerals as paradigmatic cases (Owens et al., 2018). The origins of these systems can be seen as quite similar to the origins of tallying techniques based on pegs, sticks or notches, discussed in the previous section. Body-part tallies differ only with regard to the model collection their creators have chosen: fingers and, if necessary, toes and other body parts. This choice, however, had the fundamental consequence of paving the way for the creation of number words. Hurford's proposal is that, through a gradual process of lexicalization, the names of the body parts uttered along with pairing acts in body-part tallies gave rise to numerals.

> The normal processes of language-change over a long period would lead to words which originally had bodypart associations losing these associations and becoming pure numeral, or at least counting, words. There would tend to be a phonological split, reflecting the clear semantic difference between number and bodypart concepts (Hurford, 1987, p. 82).

It is believed that a process like this took place in the formation of the Indo-European numerals. According to Mengden (2010, p. 108), "one of the least debated etymologies [of Indo-European numerals] is that of the expression for '5', **pénk^{w}e*, which is related to the concept 'finger' or 'fist'." The number words for four, eight, and ten in Indo-European languages also might have been derived from expressions related to tallies with fingers (Martínez, 1999). These and other observations "strongly suggest that, at the time of the disintegration of the individual branches of proto-Indo-European, a finger counting method had developed into a pure decimal numeral system" (Mengden, 2010, p. 108).

At the same time that a process of lexicalization may have given rise to number words, a process of re-semantification may have given rise to number concepts. Body-part tallies provide all the sufficient conditions for the emergence of number concepts according to the accounts of number acquisition we saw in Chapter 4. The first thing to notice is that, just like tallying with notches in bones or bundles of sticks, tallying with fingers and other body parts does not require previous familiarity with number concepts. Thus, anumeric or few-number cultures can develop such techniques. However, even if conceptual understanding of numbers is absent at the beginning, the point is that number concepts can arise after mastery of a body-part tallying system is obtained. This happens because body-part tallying naturally suggests the two counting principles that are absent in other methods of tallying: stable order and cardinality. Wiese (2007) explains how, starting with the principle of stable order.

> The use of fingers (and other body parts) as tallies can lead to the emergence of a stable conventional order and hence give rise to a second stage in number development: when fingers are used to represent elements of another set, they tend to be singled out successively, following the natural order of fingers on each hand. … In this order one could, for instance, start with the thumb on one hand, go all the way to the little finger, and then use the fingers of the second hand in the same way. As the differences in finger counting in modern cultures show, other orders are possible as well, of course; what is important here is that the salient order of fingers on each hand will support a convention for singling out individual fingers successively in a fixed order. … Given that body tallies are frequently accompanied by verbal tallies (namely the names for the body parts in question), a stable conventional order of fingers used in cardinality icons will lead to a stable conventional order of words (Wiese, 2007, p. 766).

Once there is a stable order of words being used regularly in body-part tallying events, the cardinality principle may be seen as a natural consequence.

> The final word in a sequence is always more salient and more accessible than the others. This leads, for instance, to 'recency effects' shown in memory experiments: the last word in a list can be better recalled and memorised than the others (probably based on a buffer in short-term memory).
>
> This leads to a prominent status of the final word that is used in an iconic cardinality representation. Once the words are used in stable order, for a set of a given cardinality, the same word will always come last and hence be particularly salient for the representation of this cardinality. This, then, will support the emergence of indexical links between individual words and sets of a certain cardinality (Wiese, 2007, p. 767).

Cognitively, these indexical links take the form of cardinal values. Once the five counting principles are being used in a body-part tallying procedure, number concepts will emerge by the same means as they emerge in today's children. According to the ANS-based account of number acquisition, the repeated association of the last word used in a tally with a certain quantity has the power of sharpening numerosity representations and, thus, produces cardinal values. In the OFS-based account, the emergence of cardinal values for numbers above three or four requires bootstrapping, which is viable at this point because all the elements necessary for it are already present in the tallying techniques. When a tallying system is in place, words/concepts for the smallest numbers are likely to already be consolidated. These words are likely to be recited along with the first three (or four) pairing acts in tallying episodes, which then go on by the recitation of body-part names (at least, this is the case in Nadahup languages, as we will see below). Thus, people who are already familiar with the number words/concepts for one, two, and, three, and are becoming familiar with a body-part tallying system, are in the right position to bootstrap. Once number concepts for numbers larger than three or four are in place, the body-part names uttered along with pairing acts are re-semanticized—they start referring to cardinal values in the context of tallying—and then the first pairs of words/concepts for numbers larger than three or four come to light.

A vivid illustration of the different stages of the processes through which number words and concepts may have stemmed from body-part tallying systems can be seen in Nadahup languages. Dâw, one of the Nadahup languages, has consolidated number words only for numbers from one to three (Epps, 2006). Speakers of Dâw can be seen as subset-knowers in that they only know numbers smaller than four. But the following Dâw sentence clearly shows that their few number words are already employed in accordance with the cardinality principle:

ãr	*nĩ̈ĩ̈*	*mutuwap*	*bok*	*çarwʉ*
I	HAVE	THREE	POT	CLAY

"I have three clay pots"

This sentence and its translation (into Portuguese) is given by Martins (1994, p. 91). In this sentence, the word *mutuwap*, whose etymology in Dâw also goes back to "rubber tree seed quantity," refers to the cardinality of a whole collection of clay pots, in full accordance with the cardinality principle. The use of small-number words consistent with the cardinality principle can provide the model from which users of a body-part tallying system could start using the last word they recite in a tallying event to refer to the cardinality of the whole collection, as happens with children in Carey's model.

The Dâw use a body-part tallying system, but one which lacks a crucial feature and thus has prevented them from developing other number words/concepts. For quantities above three, Dâw speakers use a tallying technique that Epps (2006) calls the "fraternal strategy" and Martins (1994) calls the "even/odd system." As Martins describes it, in this technique, a representation of a collection of four items is made by separating four fingers of one hand (the thumb is kept bent) into two groups of two fingers. This gesture is accompanied by the words *mẽ'n mab*, which translates into English as "has a sibling." The reason is that, in this configuration, each raised finger is accompanied by another finger, i.e., each finger has a "sibling." To obtain a representation of a collection of five items, the thumb is raised, accompanied by the words *mẽ'n mab mẽr*, i.e., "has no sibling," indicating that the thumb is alone. For six, the thumb of the other hand is placed against the first thumb to make a new pair, and then it is said again that it "has a sibling," and so on up to ten, when all fingers of both hands are grouped in five pairs and the process finishes with the words "has a sibling." In this system, one and the same expression is associated with multiple cardinalities. The phrase *mẽ'n mab* is associated with four, six, eight, and ten, whereas *mẽ'n mab mẽr* is associated with five, seven, and nine. Without a single expression for each cardinality, the corresponding number words/concepts cannot arise.[5]

Despite the fact that in Dâw the fraternal strategy did not give rise to words for four and up, in Hup and Yuhup the etymology of the word for four is clearly traceable back to this tallying technique. Importantly, Hup and Yuhup speakers no longer use the fraternal tallying system, and thus the expression "has a sibling," in these languages, is uniquely associated with four. Currently they use a base-five body-part tallying system which starts with their words for one, two, and three (Hup words were presented in section 5.2), includes a fraternal word for four, and goes on with phrases that translate into English as "one hand" (five, the base), "one other finger stands up" (six), "two other fingers stand up" (seven), "three other fingers stand up" (eight), "four other fingers stand up" (nine) and "five other fingers stand up" or "both hands" (ten). Above ten, counting goes to toes and the system becomes ambiguous. The phrases that accompany pairing acts in the interval between 11 and 14 are repeated for pairing acts between 16 and 19.

Not all of these phrases are already lexicalized and constitute real couples of number words/number concepts detached from the tallying system. Epps (2006, p. 270-271) reports that

> Numerals greater than 'five' are more likely to receive an accompanying gesture (usually a tally on the fingers), and although most Hup numerals do not depend on a gesture for their exact value to be understood, it may be difficult to label gesture as secondary (at least for those speakers who always combine the two). In the case of the ambiguous numerals like those between 11-14 and 16-19 (as with 4-10 in Dâw), however, the spoken forms are themselves not enough to indicate an exact value, and can therefore be considered dependent on gesture.
>
> Much as S. Martins (2004: 392) notes for Dâw, Hup speakers rarely make use of their numeral terms over 'five'. The expression *dəb* "many" is commonly used, and borrowed Portuguese numerals are typically preferred (particularly by younger speakers) when a specific numeral larger than 'five' is required. … Ospina (2002: 455) notes that the Yuhup numerals are frequently accompanied by gestures involving the fingers (and sometimes even the feet for higher values).

These data suggest that, in Hup and Yuhup, only the words for numbers from one to five are fully associated with consolidated cardinal values. The other "numerals" are still seen just as phrases that accompany tallies. This suggestion is corroborated by a closer look at the level of lexicalization of candidate numerals. The phrases that originated the words for four and five are already lexicalized as true numerals. In Hup, four is *hi-bab'ní*, whose etymology is analysed by Epps as "(fact)-have.sibling/accompany.nmlz." It is interesting to note that the word ends with a nominalizer (nmlz), which converts the original phrase into a noun or, more precisely, a numeral. This clearly shows that Hup speakers see the cardinal value corresponding to four as an independent concept. In Yuhup, four is *bab-ní-w'ǎ̆p*, whose etymology is analysed by Epps as "has-sibling-quantity." Here the use of the suffix *-ǎ̆p* (quantity) is what shows that Yuhup speakers are referring to the cardinal value of four,

[5]That *mɛ̃'n mab* and *mɛ̃'n mab mɛ̃r* are not number words nor associated with number concepts is further indicated by the observation that young Dâw speakers use a hybrid system where words for numbers above three are borrowed from Portuguese (Martins, 1994, p. 93).

and not to the gesture or to the idea of having a sibling. The word for five in Hup has a few variants. One variant is not a word, but the phrase "one hand" (*ˀayŭ̆p d'apŭ̆h*). But there is also another variant that displays a process of lexicalization through phonological reduction: *ˀædapŭ̆h*. The Yuhup word for five has only one variant—*cãh-põh-w'ǎ̆p*—where the suffix *ǎ̆p* shows that speakers are referring to the cardinal value of five. In contrast to the words for one to five, the phrases that accompany tallies for values above five present several variants and do not show signs of lexicalization. To give just one example, Epps (2006, p. 271) identified the following variants in Hup for the phrase accompanying the gesture for six:

cãp cob cakg'et ˀayŭ̆p	"other finger stands up one"
ˀayŭ̆p cob cakg'ĕ̆t	"one finger stands up"
cãp cob popŏ̆g	"other finger RED-big (=thumb)"

These are full phrases, with no signs of lexicalization. This observation, along with the fact that Hup and Yuhup speakers usually show their fingers when referring to six and up, and that youngsters prefer to use Portuguese numerals above five, suggests that five is the limit of proper number words/concepts in Hup and Yuhup. To reach that many, though, they must already have broken the barrier of subitizing, and they managed to do so by using a body-part tallying system.

In Hup and Yuhup, the words for numbers from six to ten are not ambiguous; all the sufficient conditions for the emergence of cardinal values seem to be in place. However, judging by the absence of lexicalization of the corresponding expressions, it seems that cardinal values for numbers above five have not emerged. Why? Ontogenetic studies have shown that the speed of development of number concepts depends on the amount of exposure to situations where numbers are used. Piantadosi et al. (2012) suggest that the transition across stages of number knowledge (from one-knower to CP-knower) is driven, to a large extent, by the input a child receives: the greater the exposure to numbers, the faster the acquisition. This has been confirmed in a study conducted with Tsimané children (Piantadosi, Jara-Ettinger, & Gibson, 2014). The Tsimané are a farming-foraging group who live in the Bolivian Amazon. Tsimané children undergo the same knower levels as English speaking children do, but between two and six years later, because number words are used less frequently in their culture. Thus, we can suppose that among the Hupd'əh and the Yuhup, hunter-gatherers who live in even smaller societies, the use of numbers above five (and the operation of tallies for quantities greater than five) is so rare that people never develop concepts for them, even if they have mastered a process which, if used more frequently, could lead them to acquire such concepts. The Hupd'əh and the Yuhup may be in the stage identified by Davidson et al. (2012) (mentioned in section 4.5) where children have generalized the cardinality principle only as a mechanical procedure, and have not yet made a semantic generalization for all the numerals in their counting list. That is, the Hupd'əh and the Yuhup may be able to realize that the final gesture made and the final phrase uttered at the end of a tallying event represent the cardinality of the whole collection, and thus can be used to denote it, even if they have not yet formed the corresponding cardinal values. This would explain why the phrases for six and up were not lexicalized. There is no point in using the suffix *-ǎ̆p* (quantity) or nominalizing the phrases to refer to a concept that they do not have. In the absence of a new concept, the phrases are not re-semanticized and speakers simply give the literal description of the tallying gesture.

If my analysis is correct, the Nadahup languages give us the opportunity to see a numeral system under construction. The building blocks are words brought in from a non-numerical context—descriptions of gestures—and a tallying procedure. The operational use of these words for tallying leads to their re-semantification, and eventually the new concepts so produced prompt the transformation of the originally non-numerical words into proper numerals. Let us take a closer look into the process of re-semantification that makes number concepts appear where they had not been before.

5.4.1 *A closer look at re-semantification*

As we saw in section 2.4, one condition for re-semantification taking place is the use of symbols to perform certain operations, instead of to denote or communicate, since this opens up the possibility of them being de-semanticized. This condition is met by the words recited during the operation of a body-part tally. In the Hup and Yuhup systems, the recited words are literally describing the gestures the operator makes with her hands (e.g., "one other finger stands up"). These descriptions can be de-semanticized because they do not aim at communicating or denoting anything. There is no point in describing for oneself each gesture one makes, or in telling others this during the production of a tally. The words recited during a tallying event are best seen as being addressed to the operator themselves, probably with the same purpose as when we subvocalize counting words: to help single out objects and/or raise fingers sequentially, so that the counter/tallier does not lose track of the one-to-one matches she is making.

There is experimental evidence that raising the exact number of fingers to represent the size of a collection can be difficult for people who have not undergone adequate training. In an experiment with deaf individuals who did not learn to count nor use a conventional sign language (but who are able to communicate with their families through a home-developed system of signs), Spaepen, Coppola, Spelke, Carey, and Goldin-Meadow (2011) showed that, although participants were able to use fingers to express quantities in certain circumstances, "they do not consistently extend the correct number of fingers when communicating about sets greater than three" (Spaepen et al., 2011, p. 3163). These home-signers learned to use fingers to represent the size of collections, but they did not learn a method to do so consistently. Without following a stable, coordinated procedure, they cannot raise the right number of fingers for quantities above the subitizing limit. The act of raising fingers to tally, which is easy for trained people, may be demanding for untrained ones.

There is also experimental evidence for the positive effect of private speech (speech spoken to oneself) on self-regulation. Research on this topic started with Vygotsky (1978), who highlighted the operational role of language in serving as a self-regulatory tool for developing children. In a more recent study, Winsler, Manfra, and Diaz (2007) tested five-year-old children in a counting task. They asked children to tap a peg with a toy hammer a certain number of times. They instructed children either to count aloud or to keep silent while performing the task, and observed that children's performance improved significantly when they counted aloud, probably because speaking the counting words aloud helped them more efficiently self-regulate their actions. It has also been observed that both children and adults use private speech more in tasks that they find more difficult, and that the use of private speech decreases as subjects become more skillful at the task (Winsler, 2009).

Taken together, the fact that singling out fingers sequentially in a tallying procedure may be difficult for people with little practice at this task (as is the case for people who are just inventing the procedure or perform it only occasionally), and the fact that private speech helps self-regulation in demanding tasks, are in line with the hypothesis that the words recited in body-part tallying serve to self-regulate tallying gestures and thus improve performance. If this is so, these words are used for operation (rather than for communication or denotation), and one condition for de-semantification is met.

A second condition for de- and re-semantification is the use of symbols in a mechanical procedure. As we saw in section 2.4, the concepts of de-semantification and re-semantification were originally introduced in the context of operative writing and formal systems. Body-part tallying techniques do not involve written language and are not like formal systems in many aspects. Nevertheless, body-part tallying and formal systems have at least two relevant similarities that make their operation automatable. First, both are regulated by clear-cut operational rules. As we saw above, tallying systems in general are regulated by modified versions of three counting principles (one-to-one correspondence, order irrelevance of pairing acts, and abstraction), and body-part tallying adds two more principles, stable order and cardinality. Second, the phrases recited and the gestures made during the production of body-part tallies, just like formulas of a formal system, are produced by clear-cut syntactic rules. For example, in the Dâw fraternal strategy, words and gestures are produced following this simple rule: fingers are raised one by one, while the phrases "has no sibling/has a sibling" are uttered alternately. The rule in the Hup and Yuhup base-five strategy says that fingers of the second hand are raised one by one followed by a phrase formed through the pattern "(1, 2, 3, 4, 5) other finger(s) stand(s) up." These simple rules, once mastered, allow for the mechanical execution of the tallying procedure, so that the operator does not have to pay attention to the meaning of the words she is uttering. The more automatized the procedure, the more "ritualized" the role of the uttered words becomes, and the weaker their association with their original meanings. In this way, the words may end up de-semanticized. In fact, linguists have proposed that number words may have originated from the ritualistic execution of a procedure involving meaningless words. The following passages by Hurford summarize what he calls "the ritual hypothesis:"

> The Ritual (or 'Eeny, meeny, miny, mo') Hypothesis is that at a stage before the development of proper numeral words, rituals exist in which sequences of words which have no referential, propositional, or conceptual meaning are recited while the human actor simultaneously points (in some way) to objects in a collection (Hurford, 1987, p. 102-103).
>
> The Ritual Hypothesis being put forward for examination here is that numeral systems arose out of counting, developed as a method of achieving a practical purpose simply and reliably, using a conventional sequence of recited words … The sequences of words used in such rituals would become interpreted numerically (Hurford, 1987, p. 104).

Hurford's ritual hypothesis is quite similar to what I am proposing here, with the difference that, in my account, the words employed in the ritual were not originally meaningless.

They were descriptions of the actions performed in tallying events that, because of their operational role in a procedure executed ritualistically (mechanically), were de-semanticized. Importantly, the ritualization/automatization of the procedure is crucial for its successful execution, since it prevents distraction. If at each step the operator stopped to grasp how many fingers she had already used, she could lose track of the tally. The procedure is more accurate if executed uninterruptedly.

Words de-semanticized by the ritualistic production of tallies provide all the psychological conditions for the emergence of number concepts. Like today's children in the earliest stages of number learning, people using a body-part tallying system are reciting a sequence of meaningless words during the execution of a procedure governed by the five counting principles. At a certain point, the de-semantification of the final word of a tallying event makes room for its association with a new meaning. In the ANS-based account of number acquisition, this new meaning is the fuzzy ANS representation of the numerosity displayed in fingers. Over time, the recurring association between symbol and numerosity will sharpen a representation of the latter and give rise to a proper cardinal value. In the OFS-based account, the new meaning associated to the re-semanticized word is a cardinal value bootstrapped from the previously formed, smaller cardinal values. Either way, as new concepts emerge and the original words are re-semanticized, the creation of new words through processes of lexicalization is encouraged.

As in other cases of re-semantification mentioned in section 2.4, here re-semantification also takes place in a mechanical procedure that helps overcome a cognitive bias or limitation. Tallying techniques enabled our ancestors and contemporary people who live in few-number cultures to evaluate the cardinality of collections with more than three or four items precisely, which they could not do by relying solely on their quantical abilities. Tallying techniques give access to information otherwise unavailable; once this information surfaces, new concepts can be formed.

Nevertheless, body-part tallying systems, even after they have given rise to number concepts and to new words, are not yet fully-fledged *verbal* counting systems. Insofar as their users still see body parts as the model collection, they are not properly counting. In verbal counting, the model collection consists of words. A fully-fledged verbal counting system will emerge only when people no longer care about mapping each element of the target collection onto a body part, realizing that, for all purposes, it is sufficient to say the ritualized words in the right order. Wiese (2003, p. 140) suggests that this last step towards purely verbal counting "would be supported by a gain in efficiency: using words instead of fingers frees the fingers of their representational task [as items of the model collection] and enables us to use them for pointing to the objects we want to represent," which could make one-to-one matching easier and more accurate.[6] Once this last step is taken, a proper verbal counting system has emerged. At this stage, though, it has become a more powerful tool, since it can benefit from linguistic resources—especially recursion—and be no longer limited in range. This is the topic of the next section.

[6]This highlights the tool-like character of counting words at their origins: they took over the function of three-dimensional objects (sticks, notches, body parts) as model collections. This is a function counting words still fulfill today.

5.5 Towards infinity

A crucial difference between tallying and counting lies in what is used as the model collection. In tallying, the model collection consists of external physical objects (e.g., sticks, notches, fingers and toes), whereas in counting the model collection is a sequence of words. Model collections of physical objects face natural limitations at relatively low levels. Although sticks and notches are in principle unlimited in number, it becomes increasingly hard to carry an increasingly large bundle of sticks or to engrave an increasingly long sequence of notches. Body parts have a limit at much lower levels. Body-part tallying systems that use only fingers and toes, like those of the Hupd'əh and Yuhup, can "count" up to 20 items only. The tallying systems used in New Guinea, which also include body parts such as wrists, elbows and shoulders, have their upper limit at about 40 (De Vries, 1994). One of the main advantages of using a verbal model collection is that it inherits from language "the liberty of convention and the power of its generative rules" (Wiese, 2003, p. 142), making verbal model collections potentially infinite.

> Because counting sequences are sets of words, we are free to choose anything we like when we want to add new elements to them, as long as we obey the phonological rules of the language we are in. Convention allows us to use arbitrary words for our purpose once the tie to fingers is loosened. And because counting words are words, we can generate infinitely many of them (based on a small set of primitive items) by employing the recursive rules our linguistic system provides: language allows us to employ recursion in our system and thereby gives it potential infiniteness (Wiese, 2003, p. 142).

A recursive system is, roughly, a system in which primitive elements are combined to obtain new, derived elements, which in turn can be combined again to obtain more derived elements, and so on indefinitely. Recursion appears in natural languages, for example, in the fact that we can obtain increasingly complex sentences by combining a limited set of simple sentences. Relying on this feature of language, an incipient verbal counting system with only a few number words can be extended indefinitely through the recursive recombination of its primitive words. Although in principle any arbitrary recursive rule for the formation of new counting words could serve this purpose, in practice the rules that guide the growth of most counting sequences around the world follow a common pattern.

The Hup and Yuhup systems are again a case in point, since they are fairly typical in this regard. As we saw above, these languages use a base-five body-part system in which the word for five translates into English roughly as "one hand," and the words for the interval from six to ten are produced according to the rule "(1, 2, 3, 4, 5) other finger(s) stand(s) up." In Yuhup, though, there is a variant of the word for ten that translates into English as "two hands" (Epps, 2006, p. 273). A portion of this particular variant of the Yuhup system displays the most typical elements and rules found in counting systems around the world.

In general, verbal numeral systems have three components: simple numerals (which can be either atoms or bases), complex numerals, and formation rules which determine how the latter are obtained from the former (Mengden, 2010, p. 25ff). Atoms and bases are called "simple" because, in mature systems, they are apparently mono-morphemic arbitrary words (this is not so in the systems of the Nadahup family, where simple numerals are compound and non-arbitrary, as we have seen). Complex numerals, by contrast, are compound words

	Number	Numeral	Rule
Atoms	1	'one'	
	2	'two'	
	3	'three'	
	4	'four'	
Base	5	'hand'	
Complex numerals	6	'one hand and one finger'	5+1
	10	'two hands'	2x5
	15	'three hands'	3x5
	16	'three hands and one finger'	3x5+1
	20	'four hands'	4x5
	24	'four hands and four fingers'	4x5+4

Table 5.1: Hypothetical Yuhup-inspired base-five system.

formed in accordance with rules that represent arithmetical operations. The most common operations across languages are addition and multiplication.

In the portion of the Yuhup system that comprises numerals for numbers from one to ten, the words for numbers from one to four are the atoms, the word for five ("hand") is the base, and the rules for combining them are addition (e.g., six is one hand plus one finger) and multiplication (ten is two times one hand). In principle, these rules suffice to generate a potentially infinite sequence of numerals. Table 5.1 presents a hypothetical system based on a generalization of this portion of the Yuhup system (using English words for ease of understanding).

Similar additive and multiplicative rules govern the formation of numerals in English, with the difference that its first base is "ten" (there are more bases, as we will see shortly), and the words from 'one' to 'nine' are the atoms (and they are truly mono-morphemic, apparently arbitrary words). Thus, the English numeral 'sixteen,' for example, is the combination of the atom 'six' and the base '-teen' (a variant of 'ten') by addition (6+10), and the numeral 'forty' is the combination of the atom 'for-' (a variant of 'four') and '-ty' (another variant of 'ten') by multiplication (4x10).

There is a further element in mature counting systems, though. Notice that Table 5.1 does not list numerals for numbers above 24. Following the same pattern of numerals for smaller multiples of five in this system—"two hands" (2x5) for 10; "three hands" (3x5) for 15; and "four hands" (4x5) for 20—the numeral for 25 should be "one hand of hands" (5x5). However, this is not the way most numeral systems proceed in situations like this. When bases would encounter themselves in the composition of a complex numeral, this encounter is preferably avoided by the introduction of a new base (Mengden, 2010). Following this practice, an arbitrary simple numeral should be introduced to mean 25, which then should be used as a new base to obtain numerals for larger numbers (see Table 5.2).

As a rule, in systems that use multiplicative rules, bases can encounter themselves at the powers of the smallest base. In decimal systems, the first opportunity for this is the numeral for 100. In English, numerals in the interval from 20 to 99 are obtained by the rule *(atom*

	Number	Numeral	Rule
Base	25	'hudon'	
Complex numerals	26	'hudon and one'	25+1
	27	'hudon and two'	25+2
	30	'hudon and one hand'	25+5
	49	'hudon and four hands and four'	25+4x5+4
	50	'two hudons'	2x25
	124	'four hudons and four hands and four'	4x25+4x5+4

Table 5.2: Continuation of the hypothetical base-five system of Table 5.1.

x base)+atom (where the addition is optional). Thus, 'seventy' is 7x10, 'eighty' is 8x10, and 'ninety' is 9x10. If this pattern were to be continued, the operation behind the numeral for 100 would be 10 x 10, which would be rendered in words as "tenty."[7] However, this encounter is avoided by the introduction of a second base: 'hundred.' This new simple numeral is then used to obtain numerals up to 'nine hundred ninety-nine.' At 10^3, a similar situation takes place and a third base is introduced: 'thousand.' By the same token, in the base-five system of Tables 5.1 and 5.2, a third base would be needed to represent 125, if the compound numeral "one hand of hudons" were to be avoided.

It is not entirely clear why most languages avoid this kind of construction and prefer to introduce new bases for the powers of the first base (or at least for some of them). The most accepted hypothesis is that this is motivated by a specific kind of counting practice, which Hurford (1987, 2007) calls the Packing Strategy. The idea is that each base is a label that "packs" a determined number of items already counted into a whole, as if they make up a new unit. The rationale behind this strategy may be expressed by two maxims: (a) count as high as you can with the words you have; and (b) minimize complexity (Hurford, 2007, p. 780-781). To illustrate how these maxims work, let us take the base-five system of Table 1 in an earlier stage when its upper limit was still five ("one hand"). It would be natural for people using this system to split up large collections into groups of five, because this (five) is the highest they can count with the words they have. Thus, to count a large collection, they could count "1, 2, 3, 4, 5," "1, 2, 3, 4, 5," …, several times. Once they have split up the collection into various groups of five, it would be natural to count the groups as units (viewing each group as "a hand"). Then, they could count how many groups of five ("hands") they have as follows: "one hand," "two hands," …, "one hand of hands." At this point, however, they have a new group of five (a group of five groups), which can again be seen as a unit. Then, maxim (b) comes in: to minimize complexity, a new word is introduced to designate a group of five groups with five elements each (i.e., to designate 5^2). This word becomes the second base of the system, from which the counting sequence can be iterated again (i.e., a new round of

[7] A word like 'tenty' did exist in Old English, where the numerals for 100, 110, and 120 continued the pattern of smaller decades. They may be rendered in Modern English as "tenty," "eleventy," and "twelvety." For the original forms in Old English see Mengden (2010, p. 90ff). Mengden also points out that Old English was very unusual in this regard, since "[t]he cross-linguistically most common strategy would be to employ the expression for the second base as an augend [i.e., a constant addend] for the numerical values from '100' onwards" (Mengden, 2010, p. 90-91).

"count as high as you can with the words you have," but now enriched with a new base, as illustrated in Table 5.2). When five groups of five groups with five elements each are obtained (5^3), they are again seen as a new unity ("a hand of hands of hands"), and then a third base is introduced to minimize complexity, and so on.

Hurford's Packing Strategy is perfectly exemplified by the counting practices of the Farem people who live in Southern New Guinea.[8] The Farem speak a language called Komnzo, whose grammar was recently documented by Döhler (2018). The Komnzo numeral system uses quite an unusual base—six—but it is completely regular in other aspects. This system is predominantly employed in a ritualized procedure for the counting of yam tubers, the most important crop for the Farem. The Komnzo system has atoms for numbers from one (*näbi*) to five (*tabuthui*), and its first base is *nibo* (6^1). Complex numerals are obtained by combining atoms and bases through additive and multiplicative rules, and a new base is introduced for each power of six up to 6^6 (*fta* (6^2), *taruba* (6^3), *damno* (6^4), *wärämäkä* (6^5), and *wi* (6^6) (Döhler, 2018, p. 94)). The practical utility of this sequence of bases for counting is made evident in Döhler's description of the ceremonial counting of yam tubers:[9]

> The counting procedure involves two men who move the yam tubers from a prepared pile. They take up three yams each, move a few meters and deposit them together in a new pile. One of the two is the designated counter and he shouts out *näbi näbi näbi* 'one one one'. This means that they have moved the first unit of six. Without pause they take up again three yams each and move them over, while the counter shouts out *yda yda yda* 'two two two'. Now two lots of six or 12 tubers have been counted. Again they pick up three yams each shouting *ytho ytho ytho* 'three three three'. The two men continue with this process until they reach *nibo* 'six' (Döhler, 2018, p. 16-17).

This is the count-as-high-as-you-can phase of this counting procedure. In this phase, they use up the entire sequence of simple numerals in their system that represent the smallest cardinal values. But notice that they are already counting groups of six. They do not need to explicitly count the elements of each group because they benefit from subitizing: each man picks up three yams at once, without needing to count them. Next comes the minimize-complexity phase:

> Now 36 yams have been counted … This amount corresponds to one *fta* or 6^2. Each *fta* is marked by putting a single yam on the side of the new pile. The two men continue until all yams have been counted, and the little pile on the side which indicates the amount of *fta* slowly grows (Döhler, 2018, p. 17).

When the whole pile of yams has already been grouped in *ftas*, the second round of counting-as-high-as-you-can commences, in which *ftas* are counted. This is followed by a new minimize-complexity phase, as Döhler describes in the continuation of the passage above:

[8] Apparently, Hurford did not base his formulation of the Packing Strategy on any description of a real counting practice. He came up with the Packing Strategy solely as an hypothetical explanation of regularities observed in numeral systems around the world. Thus, Döhler's documentation of the Farem counting method can truly be seen as a confirmation of Hurford's Packing Strategy.

[9] Döhler's video recording of the counting ceremony is impressive. It is available at https://vimeo.com/54887315.

> Next, this pile [of *ftas*] is counted in the same fashion, only that each counting yam that is put to the side, now marks one *taruba*, which corresponds to 216 or 6^3. One may continue in the same fashion. Six *taruba* make up one *damno* [which] corresponds to the amount 1,296 or 6^4. For example, one *damno* is [the] amount of yams that a man should store in order to bring his family through the year. Six *damno* make up one *wärämäkä*, corresponding to 7,776 or 6^5. Finally, six *wärämäkä* make up one *wi*, corresponding to 44,656 or 6^6. I should add that nobody in Rouku remembered the last time this number was actually reached. The recursive counting procedure gives rise to the senary [base-six] system (Döhler, 2018, p. 17).

This ritualized counting procedure clearly exemplifies the Packing Strategy and how it can give rise to the sequence of bases which are so often found in fully-fledged numeral systems across the world. The Komnzo numeral system is completely regular in this regard: it has bases for every power of 6 (up to the sixth power) and only for powers of six. Other languages may use less regular systems. There are systems that have bases that are not powers of the smallest base (such as the French system, which uses a vigesimal base for numerals such as *quatre-vingts*), and systems where there are powers of the smallest base that are not designated by a simple numeral (such as the English system, which does not have simple numerals for 10,000 and 100,000).

Be that as it may, every known fully-fledged numeral system uses a system of bases to some extent compatible with the Packing Strategy (Hurford, 1987, 2007). This has an important consequence: the potentially infinite recursive rules for the formation of numerals find a virtual limit when the counting sequence reaches the first power that should be "packed" into a simple numeral, but for which a simple numeral has not yet been designated. For example, in Komnzo this limit is 6^7. In English, the highest existing base seems to be centillion (10^{600}) (Mengden, 2010, p. 117), and therefore the limit is the next still non-named base. This does not mean that numbers exceeding this limit cannot be expressed in the language; of course they can, either by means of explicitly saying an arithmetical operation or by using non-standardized constructions in the numeral system. The point is that the standard, systemic sequence of numerals has an end at a certain point. Because of this, "linguistic numeral systems, in spite of their potentially great scope due to their recursive principles ..., must necessarily remain finite. Every numeral system of a natural language has a highest expressible number" (Mengden, 2010, p. 23). Yet, as Mengden also points out, "every numeral system is designed such that, by introducing more and more bases, it could theoretically be enlarged *ad infinitum*" (Mengden, 2010, p. 65). This means that it is not fully-fledged verbal numeral systems that are potentially infinite; it is their potential for growth that is infinite. The fact that even mature numeral systems have an upper bound will be relevant for the discussion about the emergence of the idea that numbers are infinite in the next chapter.

5.6 Conclusion

In this chapter, we saw how numeral systems may have emerged among anumeric societies. Genetically evolved quantical cognition gives us the ability to perceive discrete quantities. This ability is present in all human beings. Due to environmental and cultural factors related to practical needs, some cultures have developed, on top of it, words to designate clearly distinguishable, small numerosities, and tallying and counting procedures. When people started referring to small numerosities, they triggered a cognitive process similar to the ones that take place in today's children that culminated in the emergence of concepts for the first three (or four) numbers. As a result, the words that initially referred to transitory perceptions of numerosity, always associated with specific collections, became symbols used to express cardinal values. In a next step, pressed by the necessity of keeping track of larger quantities with the same precision provided by subitizing, people devised simple one-to-one matching techniques that led to the invention of tallying systems. When the production of tallies was accompanied by the recitation of a stable sequence of distinct words, where the utterance of each word or phrase regularly coincided with a pairing act, cognitive processes were triggered similar to the ones that take place in today's children that culminated in the emergence of concepts for numbers above three or four. As a result, the words recited during the production of tallies were re-semanticized and became symbols for numbers. Judging by the linguistic evidence mentioned above, processes like these may have taken place several times in human history, in different cultures.

If this is so, both ontogenetically (as we saw in Chapter 4) and historically, number concepts arise from the internalization of symbolic systems initially experienced as consisting of de-semanticized symbols used in procedures regulated by the counting principles. Experience with these cognitive tools is what gives us number concepts.

Chapter 6

The reification of number concepts

BASIC numerical competence emerges, both ontogenetically and historically, from the mastery of number words and the counting procedure, as we have seen in the previous chapters. In section 2.5, based on this observation, I suggested that propositional knowledge in arithmetic is nothing more than descriptive knowledge of the workings of the cognitive tools that give rise to numerical competence, namely, number words and the counting procedure. I develop this proposal in Chapter 7, but before addressing this point, we still have to examine a second step in number learning, namely, what happens when children, after already having learned to count, learn to calculate.

As we will see in this chapter, what happens in this second phase helps explain why, according to the standard reading of arithmetical statements, numerals are seen as terms referring to independent objects. Recall that at earlier stages (both in ontogeny and in history), number words and number concepts do not necessarily refer to anything. Knowing what a collection with one, two, or three objects "looks like," and knowing how to execute the counting procedure (for larger quantities) are all that is needed in order to pass the Give-a-Number task, the standard test for number understanding in numerical cognition studies. Yet, talk of numbers as if they were objects is quite pervasive, not only in mathematics and platonist philosophy of mathematics, but also in everyday contexts such as at primary school (e.g., in statements such as "the number two is even"). But where does the idea that numbers are objects, or this way of speaking of numbers as if they were objects, come from?

Here I will argue that this way of speaking comes from the common human tendency to use the same linguistic framework we use to talk about objects (prototypical objects are the three-dimensional, medium-sized things we encounter in ordinary experience) to talk about non-objects (e.g., events, procedures, properties, collections, etc.).[1] In other words, this way of thinking and speaking of numbers is due to our tendency to *reify*.

Reification, also known as *objectification* or *hypostatization*, is the act of conceiving of a non-object as if it were an object. Mostly because of the "as if" proviso, these words usually carry negative connotations. In lists of fallacies, for example, 'reification' and 'hypostatiza-

[1] I am using *object* in contradistinction to event, procedure, property, collection, thought, idea, concept, and possibly other non-objects. In the terminology I adopt here, the latter can be about or involve objects, but they are not classified as objects. An event may happen to an object, a procedure may be executed on objects, a property may belong to an object, a thought may be about an object, and so on.

tion' are alternative names for the fallacy of "misplaced concreteness," defined as the act of "[c]onsidering a word to be referring to an object, when the meaning of the word can be accounted for more mundanely without assuming the object exists" (Dowden, 2019). Despite its bad reputation, reification has undeniable cognitive benefits, as we will see below. Especially when it comes to numerical discourse and mathematics in general, reification seems to produce a major cognitive boost.

In the field of mathematics education, a number of scholars have proposed that a key component of learning mathematics is the development of the ability to reify mathematical procedures, i.e., the ability to think of mathematical operations as if they constituted a single whole, which is then taken to be a new kind of object (e.g., Dubinsky (1991); Gray and Tall (1994); Sfard (1991, 2008)). This claim is based on the observation that many mathematical statements have two possible readings, one procedural and the other objectual. For example, '2+3' can be read both as a command to do something ("add 2 to 3") or as a symbol for five. Educators claim that children start seeing mathematical operations as procedural commands, and only gradually develop the ability to see them as unitary symbols referring to an object. The ability to reify what was initially experienced as a procedure allows children to "encapsulate" the complexity of mathematical operations into a "black box," which then becomes a building block for more complex operations, which are in turn encapsulated again, and so on, making complex operations more easily manageable. Reification seems to be a cognitive prerequisite for progressing from simpler to more advanced mathematics in typical development (Sfard & Linchevski, 1994).

The benefits of reification were noticed a long time ago by Peirce. "Very shallow is the prevalent notion that this [hypostatization] is something to be avoided … The true precept is not to abstain from hypostatization, but to do it intelligently" (Peirce, 1994, §1.383). For him, hypostatization is done intelligently in mathematics.

> It may be said that mathematical reasoning … almost entirely turns on the consideration of abstractions as if they were objects. The protest of nominalism against such hypostatization … is simply a protest against the only kind of thinking that has ever advanced human culture (Peirce, 1994, §3.509).

Quine is another supporter of the utility of reification, not only for mathematics but for language in general.

> The efficacy of reification in forging links between clauses and sentences has become evident from our examples … It could be said, going a step beyond Voltaire, that if things had not existed they would have had to be invented. And indeed we have found fruitful to press our reifications beyond space and time. We posit abstract objects—numbers, functions, classes—and our natural science would be a pretty sorry affair without the loyal support of that ghostly host (Quine, 1985, p. 170).

The findings from mathematics education and numeric cognition to be discussed in this chapter lend support to the fruitfulness of reification, at least when it comes to arithmetic. As we will see, not only the idea of numbers as objects, but also the ideas of potential and actual infinity in arithmetic are likely to have emerged from successive acts of reification. What is more, by creating as-if objects, reification paves the way for the emergence of pure arithmetic. Despite Peirce's extreme optimism, though, reification also has some disadvantages, as we will also see.

This chapter is structured as follows. In section 6.1, I present Sfard's (2008) general theory of reification. In section 6.2, still following Sfard, I show how the idea of numbers as objects emerges from the reification of counting. In section 6.3, we will see that the reification of number concepts gives us a new cognitive tool—namely, numbers—which enormously increases our capacity to calculate. In section 6.4, I address the emergence of the ideas of potential and actual infinity in arithmetic by means of reificatory processes. The role of reification in the emergence of the idea that numbers are existing objects naturally suggests that numbers do not exist. In section 6.5, I address an objection that can be raised against this conclusion, according to which an argument against the existence of numbers based on the fact that our idea of number emerges through reification falls victim to the genetic fallacy. I conclude that, judging by the scientific description of our relationship with numerals and number concepts, nothing indicates that their reputed referents are really out there.

6.1 A general theory of reification

In this section, I present Sfard's theory of reification. Before starting, though, a terminological remark is in order. Sfard uses the term 'reification' idiosyncratically to refer to a specific process of objectification, which is commonly called *encapsulation* by other authors (e.g., Dubinsky (1991); Gray and Tall (1994)). In turn, she uses the term 'encapsulation' to refer to a different process of objectification. In what follows, I will adopt the standard terminology, using 'reification' as an umbrella term to cover all kinds of objectification, not just encapsulation, and 'collection' to refer to the process of objectification Sfard calls 'encapsulation.'[2] Apart from these terminological modifications, my presentation here closely follows Sfard (2008).[3]

The main tenet of Sfard's theory of reification is that reification occurs when ways of thinking and speaking characteristic of discourse about physical objects are transplanted to discourse about non-objects. We think of the physical world as constituted by enduring, clear-cut objects, which we can name and describe. We use nouns, pronouns and definite descriptions to refer to physical objects, and we usually talk about them by using their names in the grammatical positions of subject and object. Reification takes place when we apply

[2] In other words: where Sfard says 'objectification,' 'reification,' 'encapsulation,' I say 'reification,' 'encapsulation,' 'collection,' respectively. This terminological modification may seem pointless to those already familiar with Sfard's work. However, for those who are not, it could be quite confusing to use the terms 'objectification' and 'reification' as something other than full synonyms.

[3] Sfard's theory of reification is just one part of a broader theory she calls "commognition." 'Commognition' is a neologism she created by conflating the words 'communication' and 'cognition.' The core premise of commognition is that thinking is an individualized form of communication with oneself. For Sfard, all distinctively human cognitive processes result from the internalization ("individualization," in her terms) of discursive dialogical practices. Her account is inspired by Vygotsky and Wittgenstein (Sfard, 2008, p. xiii). With regard to arithmetic, commognition is in line with the evidence we have seen in the previous chapters which shows that numerical cognition results from the internalization of symbolic cognitive tools which children first experience in interpersonal space. In fact, Sfard's commognition theory originated from her long experience as a researcher in the field of mathematics education.

these cognitive and discursive structures to domains not constituted by objects, such as the domains of events, procedures, and actions. In her account, reification does not need to involve a deliberate act of postulating the existence of objects where they do not really exist; it is just a cognitive and linguistic artifice. We reify simply because it is cognitively beneficial to us, regardless of any considerations about the true nature of the "things" we are dealing with.

Sfard's ontology of objects is quite economical. For her, only perceptual objects are objects in the literal sense. She calls them "primary objects" (Sfard, 2008, p. 169). Application of the term 'object' without referring to perceptual objects is always metaphorical. In doing so, we build what she calls "discursive objects," i.e., reifications that occupy the usual places of objects in our speech and thoughts.[4] Sfard describes three mechanisms through which we build discursive objects: collection, encapsulation, and saming (in her terminology: encapsulation, reification and saming, respectively). Let us see each in turn.

Collection is the act of assigning a noun to a plurality of objects and using this noun in singular when talking about all the members of that plurality collectively.

> [Collection] is the creation of the pair <noun, specific *set of objects*>, which turns a number of objects into a single entity for any communicative purpose. For example, when we speak about the *Addams family*, we may continue and say "The Addams family *is* rich," and this is discursively equivalent to saying, in plural, "Members of the Addams family, when taken together, *are* rich" (Sfard, 2008, p. 170).

In Sfard's account, the Addams family is a reification because there is no perceptual object that could be the referent of the expression 'the Addams family,' and because this expression is used in grammatical positions typically occupied by names of physical objects. What is reified here is the plurality of family members: this plurality is composed of several objects, but by subsuming all of them under 'the Addams family,' the plurality can be treated as if it were a unit. This illustrates the remark I made above: it is not required for a reification to be in place that the speaker of the statement 'The Addams Family is rich,' besides asserting it, also explicitly claims that the Addams Family exists as an object in addition to its members. The sole metaphorical use of linguistic structures whose literal application is in discourse about physical objects is sufficient to characterize reification. This makes reification a rather common phenomenon, all the more if we consider two other ubiquitous mechanisms of reification.

Encapsulation is the act of assigning a noun to processes, operations or actions by means of which narratives about them can be told as stories about objects (Sfard, 2008, p. 170). Take, for example, the statements "I had a swim" and "Purchases are growing rapidly." In these statements, 'swim' and 'purchases' are nouns; they occupy the grammatical positions of object and subject, respectively, which are typically occupied by names of physical objects (contrast these statements with "I had a coffee" and "The trees are growing rapidly"). However, 'swim' and 'purchases' do not refer to objects, but to actions or events. The use

[4] More specifically, Sfard distinguishes two kinds of discursive objects: atomic and compound. Atomic discursive objects are not reifications; they arise in the act of baptizing a primary object. Compound discursive objects, in turn, are reifications (Sfard, 2008, p. 169-170). For the sake of brevity, I will refer to atomic discursive objects simply as primary objects, and will reserve the term 'discursive object' for *compound* discursive objects only.

of these nouns, in Sfard's account, is metaphorical. Notice that someone not familiar with the meaning of 'swim' and 'purchase' could think, based solely on the grammatical structure of these sentences and in the literal meaning of the verbs 'had' and 'growing,' that 'swim' and 'purchases' referred to genuine objects. Non-reified ways of saying the same would be "I moved in water by agitating my limbs" and "The number of events in which people acquire goods by the payment of money is growing rapidly." These statements describe actions, instead of encapsulating them into nouns. Needless to say, encapsulation saves many words. By encapsulating, we do not need to describe the whole process every time we mention it.

The third mechanism of reification is what Sfard calls "saming:" "the act of calling different things the same name" (Sfard, 2008, p. 170). Saming takes place when different primary objects are perceived as sharing defining properties, so that agents react with the same word whenever confronted with any of those primary objects. "This is the case, for example, when an interlocutor uses the word *cat* for any member of a certain family of four-legged long-tailed creatures, as opposed to using specific symbols for each catlike animal that strays into her field of vision" (Sfard, 2008, p. 111-112). In contexts such as this, 'cat' does not refer to a primary object. However, it is undeniable that single cats are perceived as similar to each other, which allows us to call all of them cats. In doing so, we assign a noun to the set of perceived similarities (as if this set constituted an object, like a universal, in platonic terms) and start using this noun as a discursive object. For example, this occurs when we say "The cat was proclaimed a sacred animal in Egypt," which is not about a specific cat, but about the species. It is practically impossible to present a non-reified version of sentences like this one, which indicates that in some cases, reification may be unavoidable.

Fortunately, we do not need to eliminate all reifications, since reification is not harmful *per se*. On the contrary, the benefits of reification to communication and cognition exceed by far its possible drawbacks, as the above examples make clear (but there are drawbacks, as we will see shortly). Sfard emphasizes that, cognitively speaking, reification can be not only beneficial but also indispensable. "We objectify because we have to" (Sfard, 2008, p. 52). Reification simplifies sentences and related cognitive tasks that are otherwise too complex, too long, or too convoluted. By reifying, we see a plurality as a unit, a complex operation as a "black box," and different things as the same. These simplifications make reified discourse less cognitively demanding, and amplify our potential to deal with increasingly more complex situations, since we can always reduce complexity by reifying some parts of discourse. Sfard also sees reification as an attempt to cope with the "incessant change" of the flux of events unfolding before us.

> Reifying is an attempt to "make the moment last"—to collapse a video clip into a generic snapshot. It is grounded in the experience-engendered expectation, indeed hope, that in spite of the ongoing change, much of what we see now will repeat itself in a similar situation tomorrow. On the basis of this assumption, reification makes us able to cope with new situations in terms of our past experience and gives us tools to plan for the future (Sfard, 2008, p. 54-55).

Thus, for example, we believe that John can deliver *the same speech* (a reification produced by means of encapsulation and saming) once more, even if in fact he is going to perform a different action. Usually, reification is a matter of idealization with practical aims. Taking into account all the minor differences between John's speeches, delivered several times, would be costly, probably useless, and could hinder planning for the future. "Considering all these gains, the importance of objectification can hardly be overestimated. Objectifying, it seems, is the very technique that gives our communication its unique power actually to shape our actions and accumulate achievements" (Sfard, 2008, p. 55).

The examples above, some of them quoted directly from Sfard, may suggest that things such as natural kinds (e.g., cats) and abstract objects (e.g., speeches) do not exist. These "things" would be mere discursive objects produced by the reification of collections, processes, and perceived similarities. In fact, Sfard's conception of object admits only tangible physical objects (primary objects), as I mentioned above. I do not want to commit to such a parsimonious ontology here, since this would demand much more philosophical argumentation than what time and space allow. However, as I will argue in the next section, at least when it comes to arithmetic, reification plays a crucial role in our idea of numbers as objects. Before moving on, though, we still need to discuss another process associated with reification—namely, alienation—and the scientific drawbacks of reification.

Alienation is a further step in reification. It is the act of alienating discursive objects from the agents who gave rise to them. Take, for example, the speech John delivered. It is possible to reify John's actions by subsuming them under "the speech," but still see the speech as something John produced. However, we can objectify the speech to a higher degree by completely eliminating John from the picture. In this case, the speech starts to be seen as something whose existence is independent of John delivering it or not—just like a non-human-made physical object whose existence is thought to be permanent and mind-independent. Linguistically, alienation appears in the use of the passive voice and in the employment of the names of discursive objects in the role of grammatical subject (Sfard, 2008, p. 50). For example, "The speech was delivered at the Parliament" alienates John from his speech, and "The purchases are growing rapidly" alienates the people who are buying more. Alienation is "an almost inevitable by-product of incorporating the newly created nouns [for discursive objects] into linguistic templates taken from discourses on material objects" (Sfard, 2008, p. 51) and it is also beneficial insofar as it simplifies communication (after all, sometimes it does not matter who are buying more). However, alienation, as well as reification, may become detrimental when we forget that certain reified and alienated collections, processes or similarities are mere discursive objects created by us.

The bad reputation reification traditionally has comes from its consequences for science and philosophy, especially when it comes to ontological questions, for obvious reasons. Sfard voices many criticisms against the reification of cognitive processes and human actions in cognitive science, which, according to her, has damaging effects on the scientific understanding of concepts such as thinking, communication and learning. I will not address these specific criticisms here, but rather concentrate on the general problem Sfard calls "ontological collapse." Ontological collapse takes place when discursive and primary objects are erroneously collapsed into one and the same ontological category of real things. In this way, mere discursive objects are believed to exist in addition to primary objects. The most obvious negative consequence of ontological collapse is the overpopulation of the ontology of the real world. Another, perhaps less obvious, negative consequence affects scientific and philosophical debates. After ontological collapse, discursive objects are seen as having essential characteristics due to reality that must be reflected in narratives about them, on pain of falsity. However, discursive objects do not correspond directly to anything in the world; many of their characteristics are a matter of definition. When this is forgotten,

> we often entangle ourselves in controversies that have every appearance of disagreements about the "correctness" of one's worldview but, in fact, cannot be resolved by appeals to empirical evidence. The mechanism that produces the illusion of factual controversy, although simple, is also mostly invisible. After objectification, we often interpret metastatements, that is, statements about discourse, as statements about the extradiscursive world … Similarly, the traditional form of dictionary definitions and of the definitions found in mathematical and scientific textbooks conceals the fact that defining is a matter of human decision about the use of words. Thus, instead of saying ***We shall call*** *a polygon a triangle if and only if it has three sides*, we say *A polygon is a triangle if and only if it has three sides*. Through the very form of sentences such as the latter we "flatten" the discursive hierarchy so that the consecutive discursive layers become like a series of transparent window panes through which all the objects—discursive (words, expressions) and extradiscursive (independently existing material objects)—seem to belong to the same ontological category of "things in the world," with their mutual relations being similarly "objective" and mind-independent (Sfard, 2008, p. 57).

Sometimes, scientists are well aware that some discursive objects are terms of art whose definition is a matter of human decision. This can be seen, for example, in the reclassification of Pluto as not being a planet, which happened after members of the International Astronomical Union voted for a definition of 'planet'—a term of art—which Pluto does not meet (IAU, n.d.). When there is no such clarity, however, scientists or philosophers may entangle themselves in controversies that cannot be solved by appeal to empirical evidence, as Sfard notices. This may be the case for philosophical debates about the nature of numbers that, from the outset, collapse numbers into the ontological category of real things. When this is done, there is no empirical evidence that can settle the debate. In order to adopt a more factual approach, we need to take a step back and consider the possibility that numbers are mere discursive objects.

6.2 Reification in numerical discourse

Number concepts are not learned through contact with any particular kind of perceptual object. Instead, number concepts result from mastering a technique for assessing the cardinality of collections with precision. This technique—counting—involves a sequence of distinctive tags (counting words) which are paired in a stable order with the elements of the collection whose cardinality we are assessing. Number concepts arise from the regular association between each tag and a specific cardinality during counting events. Notwithstanding their procedural origins, number concepts are often treated as if they represent objects of a certain kind, namely, numbers. But, given that these putative objects are not at the origin of number concepts, nor do we access them or have any information that someone has done so in other circumstances, the idea that numbers are objects is in need of explanation. In this section, we will see that this objectual view is likely to come from a cognitive process that gradually converts what were initially experienced as processes into discursive objects—in a word, reification. Peirce, who noticed the utility of reification for mathematics in general, as mentioned above, also noticed the role of reification in the objectual view of numbers long before we had the scientific understanding of the roots of numerical cognition we have today:

> In order to get an inkling—though a very slight one—of the importance of this operation [hypostatic abstraction] in mathematics, it will suffice to remember that a *collection* is an hypostatic abstraction, or *ens rationis*, that *multitude* is the hypostatic abstraction derived from a predicate of a collection, and that a *cardinal number* is [a hypostatic] abstraction attached to a multitude (Peirce, 1994, §5.534).

'Multitude' is the term Peirce uses for cardinality (Peirce, 1994, §4.175). Peirce's view is in line with the account of the reification of numerical discourse we will see in this section, except for him not mentioning that cardinal numbers emerge from the encapsulation of subprocesses of the counting procedure. This latter observation is at the core of Sfard's account.

As we have seen, children start learning numbers by learning to count collections of material objects. The learning process takes about two years in typically developing children with the "right" kind of exposure, and only at the end of this period are children able to use numerals to describe the cardinality of collections in a consistent manner. This level of proficiency is achieved when children finally understand the cardinality principle, according to which the number word used for tagging the last item in a count represents the cardinality of the whole counted collection. Before children become CP-knowers, they pass through a stage in which, if asked a "how many" question just after they have finished counting a collection, they count it again (Wynn, 1990, p. 15).

Now, in light of Sfard's theory of reification, we can identify increasing levels of reification during the process of learning to count. Subset-knowers, who understand "how many" questions merely as requests to execute a procedure (to count), show a completely non-reified understanding of numbers. They count everything again when asked "how many?" because they have not yet realized that they could refer to the procedure they have just finished by using a single word. The first step towards reification is taken when children become CP-knowers. Indeed, in light of Sfard's theory, the cardinality principle is a command to encapsulate episodes of counting. Rephrased in terms of encapsulation, the cardinality principle states that the number word used for tagging the last item in a count is a shortcut that encapsulates or names the whole operation (Sfard, 2008, p. 46-47), just like the noun 'swim' (in "I had a swim") encapsulates the action of moving one's limbs in the water. When a child becomes a CP-knower, she learns how to assign a noun—the numeral '*n*'—to the action of counting up to '*n*.' At this stage, a statement such as "There are five marbles in the box" is a shortcut to "If you count the marbles in this box, you end up with the word 'five'" (Sfard, 2008, p. 53).

That newly-turned CP-knowers do understand numerical statements in this way is confirmed by the observation that they have purely mechanical command of the cardinality principle. As we saw in section 4.5, newly-turned CP-knowers have not yet developed the number concepts which will give them full semantic understanding of numerals, which is in line with the hypothesis that, even when they use expressions like "five marbles," they still understand 'five' procedurally, as an abbreviation of counting "one, two, …, five marbles." At this stage, procedural understanding is all they need to pass the Give-a-Number task: if asked to give *n* items, the procedural understanding of numerals makes them able to "unpack" the expression "*n* items" in terms of the counting procedure ("count items up to the word '*n*'") and provide the right number of items.

But becoming a CP-knower is only the first step towards the reification of numerical discourse. Reification will not take place while numerals are not used in grammatical positions typical of nouns. Frege has famously pointed out that numerals can be used both in the positions of nouns and in the positions of adjectives. In adjectival position, numerals can be seen as expressing a property, rather than denoting an object. For example, in the sentence 'Jupiter has four moons,' the numeral 'four' describes the cardinality of the collection of Jupiter's moons. Frege, who was sympathetic to the idea that numbers were in fact genuine objects, argued that the adjectival use of numerals "can always be got around. For example, the proposition 'Jupiter has four moons' can be converted into 'the number of Jupiter's moons is four'" (Frege, 1960, §57, p. 69). He claims that, for the purpose of science, the nominal use of numerals and the corresponding view of numbers as self-subsistent objects are to be preferred. Although Frege is right in cognitive terms—a reified view of numbers is cognitively more advantageous, and therefore more adequate for scientific purposes—adjectival use precedes nominal use of numerals in ontogeny, as I will now show, which supports the claim that the idea that numbers are objects is due to a process of reification (and that numbers, therefore, are not genuine objects).

Empirical evidence shows that newly-turned CP-knowers use numerals predominantly in adjectival position. Only much later will numerals be turned into nouns and start to be used in statements such as "The number of Jupiter's moons is four" or "Four is bigger than three." In a longitudinal study with children from 14 to 30 months old in the USA, Levine et al. (2010) videotaped episodes of parent-child conversations, and analysed the use of number words in these episodes. Their data confirm that children—and also adults when talking to children—use numerals mostly in adjectival position for the description of the cardinal size of collections (like in "five little monkeys") and in the counting procedure itself. In their data, situations in which number words occupy a nominal position correspond to a very small proportion of parent-child interactions (0.3% of parent's number talk, and 0.1% of children's number talk). This reveals a fairly non-reified use of numerals at early stages of development.

Sfard points to other signs that younger children have not yet reified numerical discourse. One of them is children's refusal to use the expression "the same" when comparing collections with the same number. As we saw above, saming is a mechanism of reification where one and the same noun is applied to distinct things that bear relevant similarities. The first step for saming is the recognition that there is "something" in common between two objects or situations, so that they can be said to be "the same" with respect to this. In this way, collections *a* and *b* with, say, three objects each can be seen as having their numbers in common, and therefore as being "the same" with regard to number, but only if one thinks or speaks of number as being a "thing" shared by collections.

Sfard illustrates this point by transcribing the following dialogues between Roni, a four-year-old girl, and her parents. These dialogues took place on two different occasions when Roni and her parents had two boxes in front of them, with two marbles in each box. On the first occasion, before the transcription below began, Roni had already answered the question

"Where are there more marbles?" by replying "In none." In the follow up questions, the father intentionally tries, unsuccessfully, to make her say that the boxes have the same number of marbles.

> Father: Why? Why do you say this? [that in none of the boxes there are more marbles]
>
> Roni: Because there are two in one, and in [this] one there are another two.
>
> Father: So, this is why there is more in none of them? So, in both of them there is... what?
>
> Roni: Two.
>
> Father: And this is... more or less?
>
> Roni: Less.
>
> Father: Less than what?
>
> Roni: Than... than... than big numbers.
>
> Father: Than big numbers? That means... If there are 2 in one box and 2 also in the other, then what is there in the two boxes?
>
> Roni: Four.
>
> Father: Aha. Together, there are four?
>
> Roni: Yes.
>
> Father: And in each box there is the sa...
>
> Roni: Because it is between...
>
> Father: I see. And there is the same in each box?
>
> Roni: ...
>
> Father: How many in each box?
>
> Roni: Two.
>
> Father: Oh well... (adapted from Sfard, 2008, p. 138-139)

On the second occasion, seven months later, the child still resisted using the expression "the same" to describe two boxes with the same number of marbles:

> Mother: If there is 2 here and 2 here, in which is there more?
>
> Roni: In none.
>
> Mother: And where is there less?
>
> Roni: In none.
>
> ...
>
> Mother: And this is... more or less?
>
> Roni: It is not more and not less.
>
> Mother: Neither more nor less? So what?
>
> Roni: In the middle. (adapted from Sfard, 2008, p. 180)

Sfard points out that the problem was not with the expression 'the same' itself, since Roni was able to use it properly on other occasions, such as when she said that her friend "did the same thing" (Sfard, 2008, p. 180). According to Sfard, the point is that, although the child can perceive the similarity between the two collections, she does not see any reason to say that there is the same "thing" in the boxes in any sense, because she has not yet reified numbers.

> One may speculate that so far, the children [Roni and her friend, who also displayed non-reified language] have been using the expression *the same* while seeing something they saw before. They might be able to say, for example, that they met *the same person* on Monday and on Friday. For them, the words *the same* … implied seeing one thing at different times. No wonder, therefore, that they found no use for the expression *the same* while seeing two boxes simultaneously present alongside one another. For those whose discourse on numbers has been objectified, the "one thing" that "resides" in both boxes and thus justifies the use of the words *the same* is the object called *two*; for those, however, for whom number-words are mere sounds that people make as a part of a ritual chanting, nothing in the boxes warrants the talk about "the same things." This delicate difference clearly escaped the father, for whom his daughter's *None-has-more* and his own *They-are-the-same* were perfectly exchangeable in the numerical context (Sfard, 2008, p. 139-141).

Recall that the possibility of making statements of identity is regarded by neo-Fregeans as being the hallmark of a sortal—a concept whose instances are existing objects (Wright, 1983, p. 2). Since pre-school children refrain from making identity statements between cardinalities expressed in numerals, it may be interpreted as a sign that children at early stages of development do not see numbers as sortals yet, which is in line with young children's predominantly adjectival use of numerals, too.

Sfard calls attention to yet another sign of pre-school children's non-reified use of numerals. In making numerical comparisons, they prefer the words 'more' and 'less' to 'bigger' and 'smaller.' 'Bigger' and 'smaller' are comparative adjectives. As adjectives, they modify nouns, and are used when objects are compared (e.g., "The Maracanã is bigger than the Colosseum"). By contrast, when 'more' and 'less' function as adverbs, they modify determiners, such as adjectives, but not nouns directly (e.g., the following is non-standard English: "The Maracanã is more than the Colosseum").

> When number words are used in conjunction with *more* or *less*, like in the sentence *10 is more than 8* …, these words ['ten' and 'eight'] function as determiners rather than nouns, and this implies that the objects of the talk are sets [collections of material objects] and their elements. Indeed, it would be natural to complete such sentence to "10 marbles is more than 8 marbles." When number words are used as referring to self-sustained entities, the result of comparison is presented with the words *smaller* or *bigger* [10 is bigger than 8]… In our study, the adverbs *more* and *less* dominate the conversation, and the adjectives *big* and *bigger* appear only in the last subepisode (Sfard & Lavie, 2005, p. 252).

The non-reified use of number words begins to coexist with reified forms when children are confronted, at a later stage, with numerals in nominal position at school. This is the case of statements like "Four is even," where the numeral 'four' appears in the position of grammatical subject, and of the reading of equations such as "3+4=7." In principle, statements like these can still be unpacked in non-reified terms, but at this point children have cognitive incentives not to do so. Sfard (2008, p. 54) suggests the following "un-capsulation" for the equation '3+4=7:'

- If I have a set so that whenever I count its elements I stop at the word three,
- and I have yet another set such that whenever I count its elements I stop at the word four,
- and if I put these two sets together,

then

- if I count the elements of the new set, I will always stop at seven.

As Sfard remarks, "[a]fter this example, there is hardly need for any further argument about the merits of reified numerical discourse: The length and complexity of the 'unreified' numerical equality speak for themselves" (Sfard, 2008, p. 54). (Young children who do not yet have a reified understanding of numbers do calculate like this, as we will see shortly.)

Hofweber (2016), in the course of a philosophical investigation that has nothing to do with mathematics education, makes a similar point. He observes that an equation such as '2+2=4' can be read both in the plural and in the singular: "two plus two are four" or "two plus two is for." According to Hofweber, the plural reading betrays a less objectified view, since numerals in the plural form can still be seen as determiners ("bare determiners," in his terminology), rather than nouns. In his account, "two plus two are four" can be spelled out as "two items and two items are four items," which expresses a general fact about collections, rather than a fact about numbers as objects. Notice that the singular reading does not allow this interpretation. "[W]hen we speak in the singular and say … Two and two is four, then it seems that we are saying something about particular objects" (Hofweber, 2016, p. 130). Hofweber observes that the plural reading of arithmetic equations (in which numerals are bare determiners, therefore non-reified) would become increasingly costly, in cognitive terms, as complexity grows:

> The contexts in which arithmetical symbols are introduced [for children], and the examples with which arithmetical equations are illustrated, suggest that arithmetical equations at first express bare determiner statements. After all, teachers introduce the symbolism with explanations about sizes of collections, about there being so many marbles or cats …
>
> However, thinking about arithmetic in this way, involving bare number determiners, has its cognitive obstacles, in particular when the numbers get larger. Once we try to make calculations that are not obvious any more, and once we try to solve arithmetical problems of a somewhat greater complexity, we run into cognitive difficulties. … Anything that helps in solving arithmetical problems will be gladly adopted (Hofweber, 2016, p. 134).

At this point, children may shift from the plural reading of arithmetical equations, where numerals are bare determiners, to the singular reading, where numerals are singular terms which are seen as referring to objects. "Our minds mainly reason about objects. Most cognitive problems we are faced with deal with particular objects, whether they are people or other material things. Reasoning about them is what our mind is good at. And this is no surprise. We are material creatures in a material world of objects, and the things that matter the most for our survival and well-being are material objects" (Hofweber, 2016, p. 134).

If Hofweber's analysis is correct (and it seems to be, in light of Sfard's findings), we can expect to see children's reading of arithmetical equations changing from plural to singular, as they start dealing with more complex operations. To my knowledge, this has not yet been empirically investigated. An investigation into this question can shed further light on the hypothesis that children start their learning journey with a less- or non-reified numerical discourse and gradually move to more reified forms as they become more competent in arithmetic.[5]

According to mathematics educators, non-reified readings of arithmetical formulas are cognitively more demanding because they involve a hierarchy of operations on operations. A non-reified reading of an arithmetical formula such as '3+4' makes of it a command to operate ("add four to three"). Numerals, in turn, encapsulate subprocesses of the counting procedure ('four' is understood as the operation of counting up to 'four'). Therefore, in a thoroughly non-reified reading, arithmetical formulas become second-order operations performed with first-order operations (i.e., operations performed with the procedures encapsulated into each numeral). This is more complex than seeing arithmetical formulas as first-order operations performed with objects. That is why mathematics educators argue that reification should be encouraged throughout mathematics learning. The idea is that what is initially learned as an operation at one level has to be seen as an object at the next level, so as to become the basic units of the operations at the higher level. In the very beginning of mathematical learning, subprocesses of the counting procedure go through this transformation: first, they are encapsulated into numerals (when children become CP-knowers) and, once reified, become the units with which arithmetical equations operate. In the following stages of mathematical learning, the very arithmetical equations themselves will be reified. For example, '3+4' will start to be seen also as a symbol for seven, instead of only a command to add. In this way, '3+4' can become part of more complex operations such as 2×(3+4) (Gray & Tall, 1994; Sfard, 1991).

This observation is confirmed in studies that investigate the strategies children adopt to calculate simple additions. According to Gilmore et al. (2018, section 4.3.1), starters usually rely on the strategy called "count all:" if asked to calculate 3+4, they will use fingers or other available objects to form a group of three and a group of four, and then will count all the objects in the two groups to find out the result of the addition. In other words, they un-capsulate the numerals and count the resulting physical collections directly. More experienced children realize that they do not need to unpack both numerals: it is sufficient to unpack one of them, and count on from the other. In the "count on" strategy, they calculate 3+4 by counting "four, five, six, seven." Finally, children adopt more efficient strategies where unpacking numerals and recourse to counting are no longer needed. One of these strategies

[5]Sfard mentions that preference for singular or plural forms in numerical discourse could be a way of detecting its level of reification, but she herself did not investigate this in her sample because in Hebrew—the language in which her studies were conducted—there is no such distinction in certain situations (Sfard, 2008, p. 137, fn. 16). Hofweber himself does not cite any study corroborating his analysis (as is the practice in traditional analytic philosophy, where mere plausibility about empirical matters is regarded as sufficient).

is retrieval from memory, when the agent simply remembers the solution of an operation from previously memorized arithmetical facts. Another strategy, which more clearly shows the role of reification, is decomposition. In this strategy, the agent decomposes the addends into smaller numbers, so that she can use known facts or properties of the decimal system to calculate. For example, 70+48 can be calculated in two steps as 70+30=100, and 100+18=118. This strategy involves the idea that numbers are objects that can be decomposed and re-combined in different ways. A similar transition from strategies based on un-capsulation and counting to strategies based on memory retrieval and decomposition can be found in children's resolution of subtraction problems too (see, e.g., Caviola et al. (2018)).

When the process of reification is finished, children have an objectual view of numbers. Sfard (2010) remarks that, at the end of this process, the very meaning of the word 'number' has changed. At the beginning, for children, 'number' was nothing more than a synonym of 'number word,' but after reification takes place, 'number' becomes the name of the category of objects signified by number words. In the final step of objectification, the reified concepts named by digits and number words are alienated from their creators, and start to be seen as having properties of their own. At this final stage, it becomes more natural and straightforward to say "eight is divisible by two" than "if you divide eight by two, you get a whole number," or to say "there is no biggest number" than "if you start counting and never stop, you will never reach the end of numbers."

> By eliminating the human subject, these sentences effectively disguise the fact that numbers are discursive constructs and, as such, are human-made rather than given. With the last traces of people's agency carefully erased, even the most common arithmetical proposition … conveys the message of mind-independent existence of the mathematical object. Once reified and put into impersonal sentences, the numbers appear as to have a "life of their own." They return to their human creators disguised as exclusive masters of their own fate, whereas the participant in arithmetic discourse begins experiencing them as "happening to people" rather than caused by them, and as preexisting discourse rather than as its product (Sfard, 2008, p. 50).

This does not mean, however, that everyone becomes a platonist about numbers. As emphasized in the previous section, reification does not require the explicit postulation of the existence of extra objects in addition to ordinary physical objects. For reification to be present in numerical discourse, it is sufficient that the use of impersonal linguistic structures whose literal application is in discourses about physical objects is naturalized and regarded as an adequate way of speaking about numbers. These ways of speaking and thinking naturally erase the ties between numbers and their creators and convey the impression that numbers are genuine objects. Even so, most people, possibly including professional mathematicians, if asked about the existence of numbers, will probably not have strong opinions on this, and may even deny independent existence to them by saying something like "numbers just exist in our heads." This way of speaking of numbers as if they were objects but without necessarily committing to their existence has already been identified and named in the philosophy of mathematics. Shapiro (1997) calls it "working realism," in contradistinction to "philosophical realism." The working realist uses or accepts the mathematical consequences of numbers being objects, but does not necessarily endorse the claim that numbers exist as genuine objects.

> It [working realism] is a statement of how mathematics is done, or perhaps a statement of how mathematics ought to be done, but there is no attempt to answer the important philosophical questions about mathematics. Working realism, by itself, has no consequences concerning the semantics, ontology, and epistemology of mathematics, nor the application of mathematics in science. The strongest versions of working realism are no more than claims that mathematics can (or should) be practiced *as if* its subject matter were a realm of independently existing, abstract, eternal entities. Working realism does not go beyond this "as if." Indeed, it is consistent with anti-realism (Shapiro, 1997, p. 7-8).

Working realism is compatible with anti-realism because, although mathematical practice requires thinking of numbers as objects, it does not require the objects themselves. As Azzouni (2010, p. 36) aptly puts it, "[t]he impossibility of our being able to—as it were—'think away' our thoughts of numbers when we calculate is different from the question of whether there are such things." Shapiro's description of working realism seems to be the best characterization of the attitude most of us hold towards numbers after years of schooling. This may explain why philosophical realism/platonism is widely regarded as the most intuitive account of the ontology of arithmetic—it simply drops the "as if" proviso.

6.3 Numbers as internal cognitive tools

We saw in Chapter 5 that pegs, fingers, and other kinds of discrete physical objects, used as model collections in one-to-one correspondence, function as cognitive tools that allow us to keep track of the cardinality of collections with more than three or four items. A special version of these cognitive tools—body-part tallying systems—gave us number words, counting and number concepts. In the above section, we saw that the reification of number concepts gives us discursive objects that facilitate calculation. Calculation executed by purely procedural means, i.e., by the "count all" strategy or by means of non-numerical tallying systems, is more demanding and time-consuming. Number-based calculations, compared to these other methods, are labor-saving. Why are numbers[6] so useful?

Imagining the strategies that anumeric people could use to calculate without numbers (and without numerals) can shed light on this question. Ifrah (2000) devises a method that people using a body-part tallying system and "lacking any conception of abstract numbers" could use to calculate 16×10. In his characteristic manner, Ifrah embeds the example in a short story, which I adapt as follows:

> A group of anumeric people has recently skirmished with a rebellious neighbouring village and won. The group's leader decides to demand reparations, and entrusts one of his men with the task of collecting the ransom. "For each of the warriors we have lost," says the chief, "they shall give us as many pearl necklaces as there are from the little finger on my right hand to the little finger on my left hand." What this means is that the reparation for each lost soldier is 10 pearl necklaces. In this particular skirmish, the group lost sixteen men. Of course, none amongst the group has a notion of the number 16, but they have an infallible method: on departing for the fight, each warrior places a

[6] From this point onwards, I will use the word 'number' to refer to the discursive objects (reifications), unless explicitly stated otherwise.

> pebble on a pile, and on his return each surviving warrior picks a pebble out of the pile. The number of unclaimed pebbles corresponds precisely to the number of warriors lost.
>
> In the presence of the pile of pebbles, the man in charge collects the booty in the following manner: he says "Bring me a pearl necklace each time I raise a finger," and then he raises the little finger of his right hand, the ring finger, the middle finger, and so on until the little finger of his left hand. So without having any concept of the number 10, he obtains ten necklaces. Then one pebble of the pile is set aside, and the whole operation of bringing one necklace for each finger is repeated again, allowing another pebble to be set aside, and so on, until there are no pebbles left in the pile. In this way, our imaginary tribesman has unknowingly calculated 16 x 10, even with such limited tools (adapted from Ifrah (2000, p. 15-17)).

What this example shows is that, if we are to calculate without numbers, we need collections of physical objects (e.g., fingers, pebbles) to model the quantities we are interested in, and we need to move these physical objects around in a certain way. This means that even calculations as simple as 16×10, which numerate people who use Arabic digits can calculate very quickly mentally, must be performed externally, following a strategy quite similar to the "count all" method young children adopt. These external calculations without numbers rely on two basic elements: model collections consisting of physical items and procedures that determine how these items should be deployed or rearranged, depending on the operation one wants to perform. Notice the parallel with number-based calculations, where we have numbers and algorithms for the operations. In the transition from non-numerical to number-based calculation, numbers substitute the physical model collections, and the algorithms substitute the manipulations performed on the items of model collections. This parallel makes evident the advantage of using numbers, compared to non-numerical methods: equipped with numbers and algorithms, we do not need to execute external manipulations on collections of physical items; we can calculate mentally by using numbers and calculation algorithms instead. Even when we still need to rely on external scaffolding to compute more complex calculations, such as pencil and paper or a calculator, we no longer need to materialize quantities in the form of collections. Digits on paper or screens are seen as referring to numbers, i.e., to the discursive objects originated from reification, which substitute the physical model collections that had to be deployed in pre-numerical calculation techniques. Numbers are useful because they function as internal, "pre-made" model collections.

Consider another example, now aimed at illustrating how the emergence of numbers facilitates calculation in ontogeny. Suppose that a child who still uses the "count all" method is asked to calculate 10+4. This child already knows how to count and can understand what numerals mean, but she does not benefit from the reification of number concepts yet. In order to add ten to four, she will first deploy two physical model collections, one with ten, say, pebbles, and the other with four pebbles. Then she will unite both collections and count all the pebbles. At this early stage, the operation is performed externally, and she uses her numerical knowledge only to un-capsulate numerals in the form of collections of pebbles. Later on, when she has already reified number concepts—which happens at about the same time that she is becoming more familiar with the algorithms for operations with Arabic digits—she will no longer need to do so. Instead, she will simply calculate with numbers directly, using the relevant algorithm. At this later stage, she will start seeing numbers as if they are the collections she is "manipulating" or, more abstractly, the *sizes* of the collections she is adding. What is more, she will no longer need to rearrange the items of these ethereal model collections as she did with pebbles, since calculation algorithms already give her the result of such rearranging operations without her needing to perform them. (As we will see in Chapter 7, calculation algorithms are like shortcuts that take us directly to the outcome that would be produced if we "manipulated" the collections each number substitutes in the proper way.)

The possibility of computing with mental, pre-made model collections by means of efficient algorithms is what gives us an advantage over Ifrah's tribesmen. Numbers and the associated mental procedures that implement calculation algorithms *internalize* the external cognitive tools (based on pebbles, fingers, recited words, etc.) we started with. Recall what we saw in Chapter 2: the internalization of symbolic cognitive tools gives us new cognitive abilities. These cognitive abilities, whose implementation is described in the Triple Code Model, are what allows us to both calculate mentally and use external symbolic resources (i.e., signs that substitute the collections or sizes of collections we are interested in) to calculate with pencil and paper.

Numbers as internal, pre-made model collections also gain a new function. As with models in science, numbers, being themselves models, can be used for making predictions. In fact, Dehaene (1992, p. 6) defines calculation from a cognitive perspective as the ability "to predict by symbolic manipulation the result of a physical regrouping or partitioning act without having to execute it." For example, if ten cookies are to be distributed equally among five children, we do not need to actually distribute the cookies to find out how many each children will get; we can anticipate this by calculating $10 \div 5$, where 10 substitutes for the cookies and 5 substitutes for the children. And, of course, we can also calculate in total abstraction from any particular collection. For instance, we can anticipate the result of 10×10^{90}, regardless of it being impossible to model this operation with physical items, since its result surpasses the number of atoms in the universe. Numbers, as merely discursive objects, free us from the limitations of matter.

This freedom gives us an important by-product, namely, pure arithmetic (by 'pure' I mean arithmetic that is not primarily concerned with immediate application). Those who have already reified number concepts and are performing calculations directly with numbers can now turn to the investigation of the properties of numbers for their own sake, without envisioning application in a particular situation. Recall that the process of reification of numerical discourse makes us "working realists," which is a completely appropriate mindset for the emergence of pure arithmetic.

The applicability of pure mathematics to the physical world has been a continuous source of wonder. How can such theories, most of them developed in a predominantly a priori way, fit so aptly in the physical world? In the following passage, Russell reflects on the applicability of arithmetic to things we have no experience of:

> It seems strange that we should apparently be able to know some truths in advance about particular things of which we have as yet no experience; but it cannot easily be doubted that logic and arithmetic will apply to such things. We do not know who will be the inhabitants of London a hundred years hence; but we know that any two of them and any other two of them will make four of them. This apparent power of

anticipating facts about things of which we have no experience is certainly surprising (Russell, 1951, p. 84-85).

The instrumental origins of numbers and the way numbers allow us to make predictions, as explained above, should suffice to dispel any feeling of surprise here. We do not need to physically approach every pair of London inhabitants to know that they conform to the mental model we call two. Numbers are an internal way of modelling collections of discrete objects or, more abstractly, their cardinal sizes. Insofar as future inhabitants of London remain discrete items, a pair of them will fit the cardinal type of two, and two pairs of them will fit the cardinal type of four. Using reified number concepts, we need not make any empirical observation (besides the observation that inhabitants of London are discrete items) or any physical manipulation to arrive at this conclusion; we can draw this conclusion with our eyes closed. This may explain the impression of aprioricity in this operation. But, in fact, there is nothing especially a priori here, since we have simply applied a technique developed to cope with worldly problems to make a prediction about the future of this world.

The applications of numbers would be surprising if numbers had been created by an arbitrary act of will, or discovered by purely a priori reflection, and only later found to have wonderful applications in the world. The empirical findings we reviewed in this and in the previous chapters should suffice to show that, in reality, it was the other way round. True enough, after the emergence of pure arithmetic, new developments in number theory have found remarkable applications. But this is not surprising either. Even if the development of pure arithmetic could be carried out by purely a priori means, it builds on a corpus of techniques that were originally developed to cope with worldly problems. It is only natural that careful reflection on techniques originally deployed to cope with worldly problems can make them even more powerful.

6.4 Infinity

In the above sections, I argued that what we regard as the referents of numerals are, in fact, internal mental models (number concepts) that are seen as if they are external objects. The role of reification in this process seems to lead to a mentalist account of the nature of numbers—after all, reifications exist only in our heads, as mental concepts. In the literature on the philosophy of mathematics, mentalist accounts have been rejected on the basis of an ontological argument, which can be summarized as follows: given that human minds are finite and numbers are infinite, there are not enough mental entities to fill the ontology of arithmetic; therefore, numbers cannot be mental entities (e.g., see Giaquinto (2017, p. 2)). Related problems faced by mentalist accounts are the consequent excess of mental entities corresponding to the most used numbers—for example, there could be billions of twos, one in every mind of almost every human being alive—and the possibility of gaps in the sequence of numbers—if no one has ever thought of a certain number, this number cannot exist (Frege, 1960, §27, p. 37-38).

Despite being mentalist in a sense, an account of the nature of numbers based on reification does not run into these problems. The first point to be noted is that, according to this account, numbers, i.e., the entities that numerals apparently denote, do not exist. In certain situations, we use numerals as if they successfully denote existing objects, but in fact they do not. The idea that numerals denote numbers is only an idea, a product of reification. Numerals do not denote mental entities either; they do not denote anything (how this can be reconciled with the standard, referential reading of arithmetical statements is the topic of section 7.1). Therefore, according to this account, the ontology of arithmetic is empty (the epistemic consequences of this will be discussed in Chapter 7).

Nevertheless, the idea that infinitely many numbers exist is part of working realism. Therefore, the origin of this idea has to be explained, all the more so if in fact there are not infinitely many numbers. In the remainder of this section, I explain how the idea that numbers are infinite may have emerged by considering, as before, findings from numerical cognition and mathematics education.

Descartes famously claimed that the idea of infinity could not be accessible to finite beings, like us, unless it had been instilled in us by God, himself an infinite being (Descartes, 2008, p. 32-33). In our secular era, there is no place for God in scientific explanations, but Descartes's way of posing the problem makes clear the main difficulty in explaining the origin of the idea of infinity: how could finite beings who have only finite experiences ever come to develop the idea of infinity?

In contemporary accounts of this topic, the key to solving this problem has been the observation that it is possible to make infinite use of finite means (Humboldt (1988, p. 91); Chomsky (1965, p. v)). This possibility is taken to be behind the creative aspect of language, which allows the generation of infinitely many distinct sentences out of a limited set of words. As we saw in section 5.5, verbal numeral systems inherit this feature from language and, by means of recursive rules for the formation of words, can produce a potentially infinite sequence of number words. As we will see below, recent findings in numerical cognition suggest that the recursive rules for the formation of verbal numerals play a central role in the acquisition of the idea that numbers are infinite.

Earlier accounts, such as Carey's (2009), proposed a faster route to the idea that numbers are infinite that does not involve the recursive rules for the formation of numerals. In fact, at least in theory, this idea can emerge at the semantic level by means of the successor function. The successor function determines that, for every natural number n, its successor is n+1. Someone who knows the number one and is able to add one to any given number can, in principle, easily infer that there is no largest number. The successor function *is* a way of making infinite use of finite means.

On Carey's (2009) account, children grasp the successor function at the moment they become CP-knowers. Carey's original account, however, did not find confirmation in subsequent studies. These studies showed that children are more likely to learn a mechanical procedure when they become CP-knowers; no semantical generalization tantamount to a grasp of the successor function is achieved. This finding, first reported by Davidson et al. (2012), has been confirmed in several further studies (Cheung et al., 2017; Schneider et al., 2020; Spaepen et al., 2018; Wagner et al., 2015). Carey herself has recently admitted that, in contrast to her original proposal, knowledge of the successor function is not what causes children to become CP-knowers. The new findings suggest that it is the other way round: becoming a "mechanical" CP-knower (as discussed in Chapter 4) is a prerequisite for understanding the successor function (Carey & Barner, 2019, p. 831). To date, most developmental psychologists still maintain that children come to realize that numbers are infinite by intuitively understanding the successor function. What was not clear in earlier accounts was that children cannot do so without the help of the recursive features of numerals.

In a study with one hundred English-speaking children, Cheung et al. (2017) observed that children acquire intuitive understanding of the successor function much later than predicted in Carey's original account: around age six. In this study, a child was classified as truly understanding the successor function if she (a) was able to find the successor of every number in her count list, and (b) knew that all numbers have a successor, even numbers that are beyond the upper limit of her count list. Newly-turned CP-knowers failed to meet either requisite. In their sample, Cheung et al. (2017) identified four different levels of understanding of the successor function:

- Infinity non-knowers: children who believe that it is not always possible to add one to a number and, coherently, believe that there is a largest number;

- Successor only knowers: children who believe that it is always possible to add one, but nevertheless still think that there must be a largest number;

- Endless only knowers: children who believe that it is not always possible to add one, but even so believe that there is no largest number;

- Full infinity knowers: children who believe that it is always possible to add one and, coherently, that there is no largest number.

The existence of all these different levels of understanding shows that children's comprehension of the successor function is progressive and fragmentary. In particular, the existence of successor only knowers and endless only knowers indicates that what seems like a straightforward inference for adults—if it is always possible to add one, then there is no largest number—may be a difficult step for children. These results suggest a developmental scale with at least three stages: "[I]t seems likely that children begin by learning that every number they know has a successor, and by then generalizing this to all possible numbers, before ultimately realizing that this belief implies that numbers never end—a perhaps non-obvious entailment of an unlimited successor function" (Cheung et al., 2017, p. 31). These three developmental stages take place only after the child has become a CP-knower.[7]

In Cheung and colleagues' sample, full infinity knowers were older (their mean age was 6.28 years, contrasting with a mean age of 5.2 years for infinity non-knowers) and, importantly, were able to count up to higher numbers than children in other levels. Only children with substantial counting experience—those who could count to at least around 80—were able to find the successor of every number in their count list, and among them full infinity knowers could count to around 100. These results suggest that, contrary to what was previously thought, children need a much larger sample of number words and counting practice to generalize the successor function.

[7] Endless only knowers do not fit into this developmental pattern, since they believe that there is no highest number before they have realized that it is always possible to add one. The experimenters found only six children in this group, in a sample of one hundred children. They suggest that these children may be in fact infinity non-knowers who have randomly guessed, or been explicitly taught by caregivers that numbers are infinite before they have understood why (Cheung et al., 2017, p. 29).

That it could not be easy to infer the successor function from experience with only very small numbers had already been anticipated by Rips, Bloomfield, and Asmuth (2008), years before empirical findings confirmed that recently turned CP-knowers do not know the successor function. Rips and colleagues argued that the type of information children get from counting to only a fixed, small number is compatible with various possible inductive inferences. For example, a child who is only able to count up to ten, and who has correctly noticed that, within this limited range, the cardinal value of a number word corresponds to the cardinal value of the previous number word plus one, may still infer that ten is the highest number, or that the sequence of numbers restarts after ten (and so 'eleven' would mean one, 'twelve' would mean two, and so on). According to Rips and colleagues, there is nothing in a limited sequence of numerals that could prevent children from thinking that numbers find an end or make a loop. This is especially true of numerals for small numbers. In English and many other languages, words for small numbers are lexical primitives, i.e., arbitrary morphemes that do not display an internal structure (Hurford, 1987; Mengden, 2010). Children learn these first number words one by one, since there is no rule of formation that could tell them how the sequence continues. In this aspect, the sequence of lexical primitives for small numbers resembles other lists children are learning at the same time, such as the days of the week and the months of the year. These lists do have an end and make loops. While children know only a limited sequence of number words that are lexical primitives, they might mistakenly think that number words do the same.

However, this changes as children start learning words for larger numbers. Whereas words for the first numbers are arbitrary morphemes—called atoms—words for larger numbers are compounds of atoms and a base, following recursive syntactical rules which encode arithmetical operations, as we saw in section 5.5. When children understand these syntactical rules, they find out that the sequence of numerals is not like the sequence of the days of the week; there may be many more numerals beyond the highest numeral they have ever counted to. This may be the reason why children need to first acquire experience with counting up to higher numbers before they can generalize the successor function.

> Although the first twenty numbers in English provide little evidence for repeating structure, as children learn to count beyond 30 they gain increasing evidence for the base 10 system. After 100, they learn that the system is truly recursive, and that the entire count list from 1 to 100 can be recycled for labeling larger numbers (Barner, 2017, p. 582).

But this is not sufficient to completely dispel Rips and colleagues' concerns yet. Although the recursive syntax of numerals may suggest that number words never end, children can still think that their meanings (numbers) make loops. Barner (2017) then proposes that what prevents children from thinking that numbers make loops is the association of the counting procedure with the ANS. Even if concepts for larger numbers do not originate from ANS representations of numerosities, children can still make use of intuitions provided by the ANS to understand that numbers increase constantly.

> Specifically, if children know that the ANS represents a monotonically increasingly set of magnitudes, and that the count list is meant to explain this ordered set, then they could restrict their hypotheses regarding the logic of counting to only those models that

> result in a monotonically increasing set of precise cardinalities (Barner, 2017, p. 581-582).

All things considered, two new ingredients must be added to Carey's original proposal before children can generalize the successor function. Besides knowing the cardinality principle, (a) they need to gain experience in counting to higher numerals (about 100, in English), so that the recursive syntax of numerals can suggest to them that there are more numbers beyond the last numeral they have ever counted to; and (b) they need to establish an initial association between the series of number words and ANS numerosities.

Others have suggested that the importance of the recursive syntax of number words can be even more significant for the idea that numbers never end. Buijsman (2020) and Guerrero, Hwang, Boutin, Roeper, and Park (2020) claim that children can infer that there is no largest number by using the very arithmetical operations embedded in the syntactical rules for the formation of number words, instead of using the arguably more abstract "plus one" operation presupposed in the successor function. Buijsman (2020), analysing interviews with children reproduced in Falk (2010), observes that when children are asked to produce higher numbers, they prefer to do so by increasing the left-most component of the previously given number word, instead of simply adding one to the given number (which would be the more likely behavior if they were thinking in terms of the successor function). For example, if a child is asked to produce a number larger than "five hundred," it is more likely that she will answer "six hundred" than "five hundred and one." In this case, the realization that numbers never end could be driven by the arithmetical operations embedded in the syntax of numerals, and not by a semantic generalization resembling the successor function.

In addition to experience with higher numerals and an initial association of the sequence of counting words with the ANS, a third ingredient seems to be necessary for the acquisition of the idea that numbers are infinite: reification of numerical discourse. This is because, in order to develop the idea that there are infinitely many numbers, the child has to understand that there are more numbers than numerals she knows, and this involves conceiving of numbers as objects in their own right. Furthermore, as we saw in section 5.5, verbal numeral systems use the Packing Strategy, according to which new bases (corresponding to powers of the smallest base) have to be introduced regularly so that higher numerals can be formed. This places a constraint on the ability of users to form numerals beyond the last base they know, and makes numeral systems finite in practice. Thus, even a person who knows the full list of number words in her language needs to see numbers as objects independent of their names in order to obtain the idea that there are infinitely many numbers.

I mentioned in section 6.2 that Sfard (2010) observes that the meaning of the word 'number' changes when children reify numerical discourse. Before reification, "numbers" are nothing more than number words; after reification, numbers are seen as the objects number words refer to. This change is indispensable for children to realize that numbers never end, since number words do end. Children who have not yet made this distinction may misunderstand questions about "the highest number" as questions about the highest numeral there is, or the highest numeral they know, as Sfard (2010) notices in her comments on Falk's (2010) study.

Falk probed children's understanding of the idea that numbers are infinite through a cleverly designed competitive task, in which the aim is to say a number bigger than the number previously said by one's opponent. The participant who fails to name a larger number loses. Children who misunderstand the game as being about numerals are likely to fail, since they can run out of numerals. The following is the transcription of a 10-year-old girl (identified as "G10") playing the game with the experimenter (experimenter's numbers and remarks are within parentheses):

> G10 [in the continuous sequence of Game 1]: (120) 900 (1,000) 1,200 (2,000) 3,000 (million) million and a hundred (2 millions) 10 millions (billion). What is that? (A very large number.) I don't know, I cannot go on because I don't know any more names of numbers (Falk, 2010, p. 37).

The girl lost because she ran out of "names of numbers." Although her way of speaking suggests that she already sees numbers as being independent of their names, this did not allow her to succeed at the task. In contrast, children who correctly understand the game as being about numbers, and not about their names, and who have already understood that numbers do not have an upper bound will realize quickly that no one can win this game. An interview with another girl (G12) exemplifies this point (experimenter's question and remarks within parentheses):

> G12 [Game 1]: (Can we end the game?) No, because there is no end to numbers. (A minute ago you said that I had named a number that you didn't recognize.) This is right, but I can deal even with numbers that I don't know. Whatever you say, I'll say 2 or 3 more (Falk, 2010, p. 37).

Falk concludes:

> Conceivably, because of the important role of the label in forming the number concept (agreed among many scholars), after having made the connection, undoing the tie between a number and its name is not easily accomplished … In our three experiments, participants referred time and again to this connection, both when obstructed in perceiving the infinity of numbers because of lack of names and when succeeding in the infinity task by disengaging numbers from their names and realizing that labels can be changed or invented … The ability to acknowledge the existence of unnamed numbers proved an important key to conceiving their infinitude (Falk, 2010, p. 29-30).

Overall, the experimental results we have seen in this section show that children come to understand that numbers are potentially infinite by (a) mastering the recursive structure of numerals, (b) becoming able to generate increasingly larger numbers (be it by means of the successor function or the operations embedded in the syntax of numerals), and (c) distinguishing numbers from their names (which requires reification). The latter is especially important because, in fact, no one is capable of making *infinite* use of finite means. In practice, we make only *finite* use of finite means. "Infinite use of finite means" is only a figure of speech that points to the dissociation between what we actually have—finite lists of numerals, in the case under discussion—and how we think about what we have—that finite lists of numerals refer to an initial segment, which can be as long as we please, of the truly infinite sequence of numbers.

This leads us to the idea of actual infinity. Sfard (2010) suggests that the idea of actual infinity results from the encapsulation of unending processes that generate the potentially infinite series of numbers (e.g., the successor operation, the operation of counting indefinitely, or successive additions and multiplications). Expressions such as 'infinity,' 'infinite set of numbers,' and $\aleph_0$ are the labels that encapsulate these processes. As Falk (2010) observed in her study, in ontogenetic development, full understanding of actual infinity comes much later. Falk found that children start understanding basic aspects of the idea of actual infinity from about age eight, but their conception is incomplete and it may remain incomplete even in adult life. This confirms the expectation, suggested by the history of the topic, that the conception of actual infinity is more removed from commonsense. (I refer the reader to Pantsar (2015) for an account of the emergence of the mathematical conception of actual infinity that is in line with the findings in numerical cognition and mathematics education reviewed here.)

6.5 The non-existence of numbers

We have seen in the previous chapters that, both ontogenetically and historically, the emergence of number concepts is not related to acquiring familiarity with a class of objects (be they spatiotemporal or not), but to learning some words and mastering techniques such as tallying and counting. In addition to this, in this chapter we have seen that the idea that numbers are existing objects seemingly emerges from a process of reification of these techniques. All things considered, we are naturally led to suspect that numbers conceived of as genuine objects do not exist. We speak and think as if they do, we need to do so due to cognitive reasons, but the idea that numbers are existing objects is a mere product of reification. The conclusion that numbers *do not* exist seems to follow naturally. This description of the cognitive processes underlying human numerical competence suggests nominalism.

But why should a story about how children acquire numerical ideas matter for an investigation into the existence of numbers? Is this not a case of genetic fallacy? Genetic fallacy is a fallacy of irrelevance: in certain circumstances, a genetic account of how some idea came about may be irrelevant for the cogency or truth of the idea. An accusation of genetic fallacy here could run as follows:

> The fact that children form the idea that numbers are objects through a process of reification is irrelevant to the claim that numbers really exist. The claim that numbers exist is justified by other reasons, linked to the truth and objectivity of arithmetic, which have nothing to do with how this idea comes about in our minds. Even more so if we take into account that numbers, if they exist, are outside of space and time, therefore completely inaccessible to us. For obvious reasons, numbers themselves cannot play a causal role in children's numerical education. These psychological processes which culminate in the "reification" of number concepts may be exactly the ways through which our minds are opened to the self-existing realm of numbers.

There are two problems with this objection. The first is the assumption that existing numbers have to be postulated in order to account for the epistemic properties of arithmetic, such as truth and objectivity. As I will show in Chapter 7, this is not so. What truly justify our numerical beliefs and explain their truth and objectivity are their connections with the practices that give rise to them. Since for now I cannot advance on this topic—since it requires a whole chapter—let me focus on the second problem.

The second problem is that this objection only holds from a dogmatic point of view. It assumes from the outset that numbers are abstract objects outside of space and time, and relies on this assumption to explain why we should not expect to find numbers having any causal role in this world, in particular in children's numerical education. As I argued in Chapter 1, platonism makes the hypothesis that numbers exist (as platonic entities) irrefutable. In fact, the existence of numbers as platonic objects is compatible with everything I presented here. It may really be the case that we come to believe that numbers are objects by means of a process of reification, and, by happy coincidence, the final results of this process are beliefs that correspond nicely to a description of the real, but inaccessible, realm of non-spatiotemporal numbers. We will never know. But this conclusion does not recommend belief in the existence of numbers nor agnosticism. A point famously made by Russell about belief in the existence of God fits this situation nicely.

> Many orthodox people speak as though it were the business of sceptics to disprove received dogmas rather than of dogmatists to prove them. This is, of course, a mistake. If I were to suggest that between the Earth and Mars there is a china teapot revolving about the sun in an elliptical orbit, nobody would be able to disprove my assertion provided I were careful to add that the teapot is too small to be revealed even by our most powerful telescopes (Russell, 1997, p. 547-548).

Russell grounds his atheism, rather than agnosticism, in the observation that the hypothesis that God exists is irrefutable by definition. The same attributes that make God irrefutable make him undetectable, in the same way that the minuteness of the teapot makes its existence undetectable but also irrefutable. This should place the burden of proof with the dogmatist. If Russell does not provide any other reason to support his claim that there is such a teapot orbiting the Sun—he does not show that someone has launched it into orbit, for example—then we are justified in emphatically denying its existence. By the same token, if we do not have any other reason to think that God exists—science has explained most of the phenomena that God was believed to be involved in—then the skeptic is justified in being atheistic, rather than merely agnostic. It is not difficult to see that the non-spatiotemporality of platonic numbers plays the role of the minuteness of Russell's teapot, making platonic numbers undetectable and irrefutable at the same time. True enough, the platonist provides other reasons for the existence of numbers, linked to the epistemology of arithmetic. However, once platonic numbers have been explained away in the epistemology of arithmetic—which is the case if my arguments in Chapter 7 are sound—we are justified in emphatically denying the existence of platonic numbers.

Another point that can be adduced to show the relevance of a genetic account of number beliefs is the fact that realists themselves have adopted this approach. The claim that numbers are abstract objects outside of space and time makes numerical knowledge a mystery, and therefore realists feel the need to explain the origins of numerical beliefs so as to explain how we can acquire the correct beliefs about a non-spatiotemporal reality. For example, Shapiro dedicates several pages to a (mostly speculative) explanation of how humans could obtain knowledge of *ante rem* structures through ordinary experience with physical patterns (Shapiro, 1997, p. 109ff). As Shapiro points out, "[t]he realist … owes some account of how a physical being located in a physical universe can come to know about *abstracta* like mathematical objects" (Shapiro, 1997, p. 110). What Shapiro does not do is investigate the real psychological processes through which those beliefs come about. And, as the good practices of scientific methodology recommend, such a scientific investigation should not be conducted in order to confirm a point dogmatically believed to be true in advance. This is why a scientific investigation of the origins of numerical beliefs must be open to the possibility that these beliefs have nothing to do with *abstracta* (in the sense of being outside of space and time) at all. A genetic account of our numerical ideas in the spirit of the "reverse engineering" proposed in the introduction to Chapter 2, which takes a neutral stance on the existence and nature of numbers, seems to be the best way to undertake a non-dogmatic scientific investigation of this topic. As we have just seen, this reverse engineering ended up revealing that numbers are mere discursive objects and do not have real existence. To nail down this conclusion, the only thing still lacking is an explanation of arithmetical knowledge in the absence of existing numbers, which I will provide in the next chapter.

6.6 Conclusion

The sequence of events in cognitive development and history we have been seeing since Chapter 4 reverses the traditional schema according to which first there were the eternal numbers, then number representations in the mind, and finally numerals (created and used by those who already knew numbers). The sequence of events we saw reveals that it is actually the other way around: first there were numerals (created and used by people who did not yet know numbers), then number concepts, and finally numbers (discursive objects, the reputed referents of numerals and number concepts). Those eternal numbers of the traditional schema apparently do not exist. The thought that numbers are objects seems to be cognitively indispensable, but nothing indicates that they are really out there.

Chapter 7

Back to the philosophy of arithmetic

The role of reification in the emergence of the idea that numbers are objects suggests that numbers, conceived of as genuine objects, most likely do not exist, as argued in the previous chapter. What is needed to assert more conclusively that numbers do not exist is an account of the semantics and epistemology of arithmetic in which the nonexistence of numbers does not compromise the standard reading of arithmetical statements nor the epistemic attributes traditionally ascribed to arithmetic.[1] It has been claimed that nominalistic accounts of arithmetic must either give up the referentiality of numerical statements or their truth. Wright, for example, claims that the neo-Fregean argument for the existence of numbers "must succeed unless either the apparent singular terms of arithmetic do not really function as such or the apparently true 'appropriate' contexts in which they feature are not really true" (Hale & Wright, 2001, p. 154). More generally, platonists have argued that the existence of numbers must be postulated if an account of the referentiality, truth, objectivity, necessity, and apriority of arithmetical statements is to be tenable.

Against these views, in this chapter I submit a nominalistic, empirically informed account of the semantics and epistemology of arithmetic that makes it possible to maintain the referential reading of arithmetical statements and explain why and in which senses they are true, objective, necessary, and a priori. All the elements necessary for this have already been discussed in the previous chapters. We have already seen that arithmetical beliefs are shaped by counting and calculating practices, rather than by a class of objects. What we will see in this chapter is that it is these practices that constitute the objective subject matter that makes arithmetical statements true, objective, necessary, and a priori. The epistemic role traditionally ascribed to numbers as existing objects is played by the procedures encapsulated within numerical statements. This is the core idea to be elaborated here.

This chapter is structured as follows. In section 7.1, I introduce a Peirce-inspired triadic account of reference which allows the treatment of numerals as referring terms in the absence

[1] There may be philosophers who do not mind if it turns out arithmetical statements have to be read non-literally, or if they end up to be false. This is not my case. Furthermore, there is nothing in the scientific description of numerical cognition that we have seen in the previous chapters that could suggest an alternative reading for arithmetical statements or their falsity. On the contrary, the objectual reading of arithmetical statements is not only real as a human practice but also encouraged by mathematics educators, as we saw in Chapter 6.

of extant numbers. In section 7.2, I explain the truth of arithmetical statements in virtue of the processes they encapsulate. In section 7.3, I investigate the normative and descriptive aspects of arithmetic, and explain in which sense it is necessary, objective, and a priori. In section 7.4, I contrast some key aspects of my nominalistic account of arithmetic with similar accounts: Wittgenstein's, Kitcher's, and fictionalism. In section 7.5, I summarize the argument for the nonexistence of numbers I presented in this dissertation and exemplify which kind of empirical counter-evidence might refute it.

7.1 Separating semantics from ontology

In the standard platonist account, symbols for numbers function as names that designate existing objects. Both symbolic systems and the objects they designate are usually viewed as freestanding entities, whose links to human beings are thought to be irrelevant to semantic inquiry. As a result, the relationship between a numeral and the number it designates is seen as *dyadic*: a relation between the symbol and the object only. But, if we ask how this relationship was established, we have to add a third element. The symbol 'five,' for example, is a linguistic convention, and as such it designates five only because most English speakers associate 'five' with the number five. That is, *people* are what establish the relationship between numerals and numbers. Observations such as this led Peirce to argue that meaning is, in fact, a *triadic* relationship. For Peirce, a symbol (in his terms, a sign) consists of a triadic relationship between the signifier (e.g., pixels on a screen), the referent of the signifier, and the "interpretant." The interpretant is the understanding an agent has of the signifier as referring to something else. In Peirce's words, the interpretant is "the effect [of the signifier] upon a person" (Peirce, 1998, p. 478). It is the mental component of meaning.

The Peircean triadic approach is as compatible with platonism as the more common dyadic approach. In fact, the dyadic approach can be seen as a simplification of the triadic approach. After all, it is undeniable that minds play an essential role in establishing relations between symbols and their meanings, but from this it does not follow that all scientific or philosophical investigations of meaning should take the mental part of the phenomenon into account. In the analytic tradition, semantic inquiry has predominantly disregarded mental aspects and assumed a dyadic approach. However, although the dyadic approach may suffice for certain purposes, it is not a simplification without consequences.

First, it has to be noted that the dyadic and triadic approaches differ in their conditions for meaningfulness. In the dyadic approach, a sentence wherein referring terms occur can be meaningful only if its constituent terms successfully denote an object. In the triadic approach, in turn, there is no meaning without a mind, since it is the mind that establishes the relationship between symbols, mental contents, and the world. These different conditions for meaning lead to different ontological consequences. In the dyadic approach, the fact that a statement is meaningful implies (at least prima facie) the existence of the referents of its constituent terms. In the triadic approach, this implication is not direct, since a referring term can fail to denote, but still be meaningful. The presence of an intermediate between symbols and the world—a mind—in the triadic approach opens up the possibility of referring terms having meaning without denoting because their meanings can be confined to the activation of a mental concept at the intermediate level. The dyadic approach conflates semantics and ontology. The triadic approach allows for a separation between semantics and ontology.

The semantics of apparently empty names poses a problem for dyadic approaches. The classical examples are names of fictional characters. How can a sentence such as 'Sherlock Holmes lives at Baker Street' be meaningful and putatively true if 'Sherlock Holmes' does not denote anything in reality? There are several proposals on how to accommodate apparently empty names in a dyadic semantics. Platonists deny that apparently empty names are really empty and postulate the existence of a non-spatiotemporal reality where the referents of these names are to be found. Nominalists can opt between translating empty names into definite descriptions, using a free logic, or claiming that it is possible for a name to stand for a non-existing thing. Another option for nominalists is to reject the dyadic approach altogether, in favor of a triadic approach. This is the path I will take here.

A decisive factor for my preference for a triadic approach over dyadic alternatives is that the former is more faithful to the phenomenon I aim at modelling. After all, meaning is a "human thing." True enough, a scientific or philosophical account of a phenomenon can disregard some aspects of it if they are not essential to the theory. However, it is tempting to think that the difficulty the dyadic approach has in accounting for the semantics of empty names points to the need to take additional aspects of the phenomenon into account. The ease with which the semantics of empty names is accounted for by the inclusion of an intermediate level between names and their optional worldly referents suggests that the missing aspect in the dyadic approach is the human factor.

The inclusion of minds in the semantics of mathematical expressions has been rejected in the face of apparent difficulties. Frege, for example, has famously objected to this due to concerns regarding identity and objectivity. According to him, if the content of the Pythagorean theorem were an idea in the mind, then "we should not really say 'the Pythagorean theorem', but 'my Pythagorean theorem', 'his Pythagorean theorem', and these would be different" (Frege, 1997, p. 336). Here, I will not offer a general solution to this problem. However, at least when it comes to the semantics of numerals and arithmetical expressions, we will see that these concerns are unwarranted. Before addressing the semantics of numerical statements, though, let me introduce some conceptual distinctions, mostly inspired by Azzouni (2010) and Crane (2013), which will help in this task.

Azzouni (2010) and Crane (2013) provide two similar accounts of the meaning of empty names. They are concerned both with preserving the referentiality of empty names and, at the same time, accounting for their meanings without needing to connect them to existing objects. To this end, they both distinguish between two kinds of semantic relationships that names can have with their referents. Let us look at their proposals in turn.

For Azzouni, both empty and genuine names *refer* to something, but in two different senses of 'reference.' In the first sense, which he calls *reference*^{r}, there is a genuine relationship of denotation between a name and the object it designates in the world. Genuine names refer^r. In the second sense, which he calls *reference*^{e}, there is no relation of denotation. Empty names refer^e. For Azzouni, reference^e is not a relationship at all. "Instead, [reference^e is a characterization] of certain terms (names, quantifiers, etc.) when such play a certain role in discourse: have grammatical and semantic roles in sentences indistinguishable from otherwise referential^r terms" (Azzouni, 2010, p. 24). In other words, for Azzouni, reference^r

establishes a real relationship between a genuine name and an object in the world, whereas reference^{e} is an intralinguistic phenomenon typical of empty names. In Azzouni's account, the observation that a name such as 'Pegasus' refers (more precisely, refers^{e}) solely means that the name 'Pegasus' fits in

> the name-schemata '___' refers to ___, one that applies to every name by virtue of sheer grammatical role. That this name-schemata can be appropriately applied to a name doesn't require a metaphysical state of affairs in which that name is related to an object—in any sense of 'object' that attributes a metaphysical status to such (Azzouni, 2010, p. 24).

For Crane, all names are about "intentional objects" (or "objects of thought"), but only genuine names refer to objects in the external world. Aboutness, as defined by Crane, is a relationship between words and thoughts only, whereas reference is a relationship between words/thoughts and the world. In Crane's account, 'Pegasus' does not *refer* to Pegasus, since this winged horse does not exist, but 'Pegasus' is *about* Pegasus, conceived of as an intentional object. Crane defines intentional objects as "those things in the world which we think about; or those things which we take, pretend, or otherwise represent to be in the world; or which we merely represent in thought" (Crane, 2013, p. 4). According to this definition, intentional objects can have an external counterpart or not. All names are about intentional objects; those which are about intentional objects that have an external counterpart also refer; those which are about intentional objects without an external counterpart, do not.

Azzouni's and Crane's distinctions are not merely terminological. They point to different parts of a name's semantic relationship with the mind and the world in a triadic semantics. Consider the three-level semantic schema depicted in Figure 7.1A. Once we have introduced the intermediate mental level, the relationship between a name or noun and the world is divided up into two segments: first, the relationship between the symbol and the mind; second, the relationship between the mind and the world. It is easy to see that Azzouni's reference^{e} is a relationship of reference where only the first segment of the semantic relationship is present, whereas reference^{r} spans the two segments.[2] By the same token, Crane's aboutness corresponds to the first segment of the semantic relationship, whereas Crane's reference spans the two segments.

Following Azzouni and Crane, I will use different terms to refer to the different segments of the semantic relationship. I will reserve 'reference' to designate the first segment (reference^{e} and aboutness, in Azzouni's and Crane's terminologies, respectively), and 'denotation' to designate the second segment (reference^{r} and reference, in Azzouni's and Crane's terminologies, respectively). The distinct uses I am making of 'reference' and 'denotation' are not intended to capture a real distinction in the ordinary meaning of these words. Even so, my use of these words does benefit from a slight difference in their ordinary meanings. One of the dictionary definitions of the verb *to refer* is to direct someone's attention or thoughts

[2] As mentioned above, Azzouni claims that reference^{e} is not a relationship at all, but an intralinguistic characteristic of names. Azzouni seemingly takes language abstractly, and hardly speaks of meaning in terms of mental contents, although it seems that he admits that, ultimately, the meaning of empty names is confined to the mind (Azzouni, 2010, p. 236 and p. 229 fn. 17). Here, I am freely interpreting Azzouni's account. In my view, the distinction between reference^{e} and reference^{r} becomes much more natural when conceived of in terms of a triadic approach where minds are the protagonists of semantic relationships.

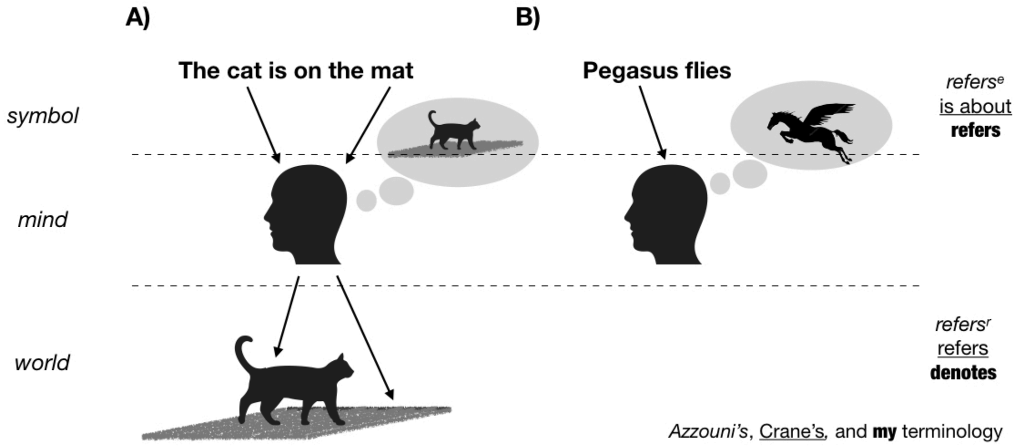

Figure 7.1: In (A), the agent interprets the sentence as being about the world. In (B), the agent interprets the sentence as being about an intentional object lacking an external counterpart. The arrows indicate the segments of the semantic relationship mediated by the mind. In the first segment, nouns/names refer to mental contents. In the second segment, depending on how the agent interprets the relationship of the sentence with the world, nouns/names may or may not denote objects in the world.

to something that can be, for example, information. By contrast, the verb *to denote* conveys more explicitly the idea of a symbol standing for something. For instance, a superscript number placed after a word in a text *refers* the reader to a footnote; its function is not to denote the footnote, but to direct the reader's attention to a side comment. Analogously, we can think of a name/noun in a sentence as primarily directing the attention of its addressee to a mental content. Denotation, in turn, requires from the addressee the establishment of a link between the mental content referenced by the name and an external object.

Using this distinction, we can say that a name can refer to something (in the mind) without denoting a real thing (in the world). For example, in Figure 7.1B, 'Pegasus' refers to the idea of a flying horse (a mental content) without denoting a real animal in the world. Denotation, just like reference, is a semantic relationship, but it is not essential for names to have meaning. A statement can have meaning regardless of its terms successfully denoting or not. This distinction separates semantics from ontology. Semantics is concerned with how cognitive agents interpret symbols; ontology with what exists. Reference, as construed here, is an intra-semantical relationship. Denotation, as construed here, is an extra-semantical relationship that connects symbols and mental contents with the external world. Since this connection is not necessary for meaning, in triadic approaches of meaning, ontological commitment is not dictated by the logical form of sentences, but by the intentions of the speaker. If the speaker intends to denote objects in the world by using a name or a quantified sentence, then, in this case, she is ontologically committed to the existence of those objects. Otherwise, she may be operating only at the mental level, referring to and quantifying over "intentional objects," in Crane's terminology.

> Quineans see ontological commitment as held primarily by theories. An alternative is to see ontological commitments as held ultimately by thinkers: thinkers commit them-

> selves to the existence of things when they frame their theories of the world. Without thinkers, after all, there would be no theories of the world. This suggests that the phenomenon of 'ontological commitment' should ultimately be explained in terms of the mental states of thinkers (Crane, 2013, p. 48).

These considerations lead to a reconstrual of what a model-theoretic semantics of a formal language implies in ontological terms. Following Quine, it is commonly assumed that the objects that have to be included in the domain of a regimented theory so as to make its statements true constitute the ontological import of that theory. In other words, all the objects in the domain of the theory have to exist (at least according to the theory). But when we take into account the distinction between reference and denotation, there may be objects in the domain of the theory that do not denote anything in the world. These may be the referents of singular terms or the values of bound variables that refer to intentional objects that lack worldly counterparts. In this case, we cannot view the model of a theory as a final representation of its ontological import. Since what distinguishes mere reference from denotation are the intentions of the speakers (or the intentions of the authors of the theory), in order to find out the ontological import of a theory, we first have to ask of each of the objects in its domain whether it is intended to represent an extant object or an intentional object without an external counterpart. In not doing so, we risk committing ontological collapse (Sfard, 2008), that is, collapsing merely discursive objects (such as reifications) and genuine objects into one and the same ontological category of real things.

True enough, often the intentions of the speaker or the proponent of a theory are not clear even to herself. As we have seen, when reification is the process behind the creation of a discursive object, the agent speaks as if the name of the object really denotes something, without needing to be aware of the as-if attitude. This means that the investigation into the ontological import of theories is not a trivial task. In certain cases—as in the case of arithmetic—a scientific investigation of what agents are in fact talking about, in spite of what they are apparently talking about, is required to distinguish non-denoting from denoting terms in their discourse. This is what I have tried to do in the previous chapters.

Model-theoretic semantics is dyadic and, therefore, it is not able to capture the distinction between denoting and non-denoting terms. Both denoting and non-denoting terms refer; model-theoretic semantics can be seen as capturing only the referential aspect of names. Thus, if model-theoretic semantics is to be integrated in a triadic approach, domains and interpretation functions must be placed at the intermediate level. Figure 7.2 illustrates how this can be done. The model of a sentence (or theory) represents how agents "read" the sentence (or the sentences of the theory), following the usual syntactic and semantic conventions, in which names/nouns are referring terms. If the sentence/theory is intended to be about the world, then relationships of denotation are established, but these are not captured by the model. Only in cases in which all terms and quantifiers denote, can the mental and worldly level be collapsed, and then the model can be seen as representing the sector of the world the sentence/theory speaks of directly. These are the cases in which the dyadic approach provides a correct, yet simplified, semantic account. When denotation is not present, though, the model only represents how the sentence/theory is read by the agent, leaving unspecified how and whether the sentence is connected to the world.

A sentence whose referring terms do not denote can still be talking about the world.

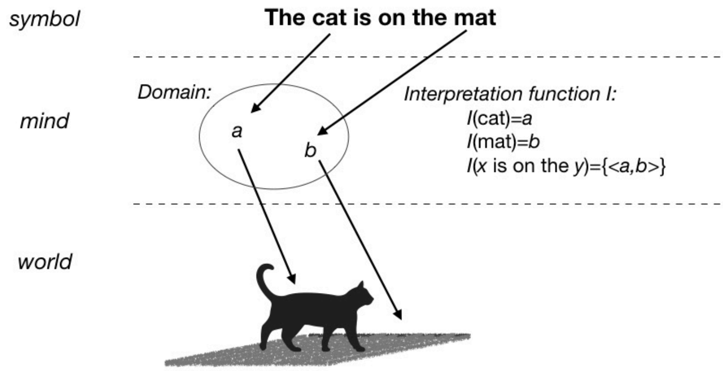

Figure 7.2: In a triadic semantics, the model of a sentence (the set-theoretic structure consisting of a domain and an interpretation function) stands for its interpretation at the mental level.

Reference in the first segment of the semantic relationship does not necessarily require denotation in the second segment if a relationship with the world is to be established. Recall that in reificatory discourse, we use syntactic and semantic resources typical of discourse about objects to speak of existing things in the world that are not objects. We create discursive objects by reifying pluralities, similarities, and procedures. Names or nouns of discursive objects created by one of these processes of reification will not denote, but even so they will have worldly counterparts—though their external counterparts are not objects, but pluralities, similarities, or procedures. To illustrate this point, let us turn back to one of Sfard's examples of reification we saw in section 6.1, the sentence (a) 'The Addams family is rich.' Assume that the Addams are real individuals, rather than the homonymous famous fictional characters. In this case, 'The Addams family' refers to a discursive object created by collection. This expression does not denote an object in the world (assuming that only individuals exist), but nevertheless it is incorrect to say that the expression 'The Addams family' does not pertain to the world at all. The point is that 'The Addams family' does not denote, but rather *collects* several objects in the world. It can be said that its function with respect to the world is to subsume several objects (individuals, in this example) under the same name. Figure 7.3 illustrates this point.

In this way, sentences wherein non-denoting names occur can still be true about the world. 'The Addams family is rich' will be true if and only if its members, taken together, are rich. As a general rule, it can be said that the truth conditions of a sentence wherein names of discursive objects produced by collection, saming, or encapsulation occur are the truth conditions of its un-reified version. Azzouni and Crane offer similar accounts of the truth conditions of sentences wherein empty names occur, as we will see next.

Following his distinction between reference^{r} and reference^{e}, Azzouni distinguishes two kinds of worldly facts that should be looked into in order to determine the truth value of a sentence. Sentences whose names and quantifiers refer^{r} are made true or false by the properties of the objects they reference^{r}; following standard practice, Azzouni calls these objects the *truth makers* of these sentences. Sentences whose names and quantifiers refer^{e} have no truth makers, but there may be other worldly facts they describe that make them true or false. In Azzouni's terminology, these other worldly facts are the *truth-value inducers* of these sentences.

> [A truth-value inducer] is to be contrasted with a "truth maker." The truth-value inducers are those factors in the world (objects and relations among such) that—dovetailing with our expressive and inferential needs—force truth values on the sentences we use. In cases where a sentence contains terms that refer^{r}, the relata of such terms and relations among such can be described as truth-value inducers that are truth makers. But where the terms in a sentence refer^{e}, the truth-value inducers are other things (and relations among such) that the terms in such a sentence are neither about^{e} nor about^{r}. "Mickey Mouse was invented by Walt Disney," for example, has among its truth-value inducers, certain objects (Walt Disney, drawings, etc.) that induce its truth value. Mickey Mouse isn't among these (Azzouni, 2010, p. 25).

Azzouni views truth-value inducers as a genus of which truth makers are a species. All sentences that are intended to be about the world have truth-value inducers. Sentences wherein referring^{r} terms and quantifiers occur have truth-value inducers which are truth makers, as generally held. Sentences wherein empty names or quantifiers occur have other worldly factors, which are not truth makers, as their truth-value inducers. Using Azzouni's terminology, we can say that sentences wherein names of discursive objects produced by reification occur have as their truth-value inducers the pluralities, similarities, or procedures these names reify. In the example of Figure 7.3, the members of the Addams family are the truth-value inducers of the sentence 'The Addams family is rich.'

Following Crane (2013), we can call the process of explaining the truth of a sentence with empty names in virtue of its truth-value inducers *reductive explanation*. "A reductive explanation is an explanation of why truths of a certain kind are true, where this explanation need not appeal to the entities apparently invoked by the truths to be explained" (Crane, 2013, p. 124). Crane gives an example of reductive explanation that is very illustrative for our purposes here. In the sentence 'The average family in the UK has 1.9 children,' the definite description 'the average family in the UK' is empty. Even if we assume that families exist, the average family certainly does not. Obviously, the presence of this empty expression should

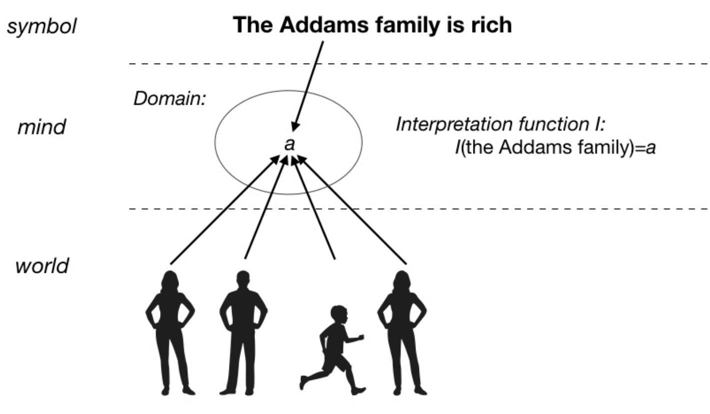

Figure 7.3: The discursive object *a*, the referent of 'The Addams Family,' subsumes a plurality of primary objects.

not be seen as making this sentence false. "[E]veryone understands that for this to be true, there need be no such entity as the average family which has this mysterious number of children. All that needs to be true is that the number of children divided by the number of families is equal to 1.9 (or something near enough)" (Crane, 2013, p. 122). In other words, the calculation process that produced this figure is the truth-value inducer of this sentence. In fact, using Sfard's (2008) account of reification, we can say that the average family in the UK is a discursive object that encapsulates a bunch of statistical calculations about families in the UK. The truth conditions of this sentence are given by its un-reified version, wherein the process of calculation is made explicit. The original sentence will be true if and only if the input data described in its un-reified version is true regarding families in the UK and the calculation of the average is correct.

Sentences wherein empty names that are names of discursive objects occur have clear links to the world which can make them true or false, in spite of the fact that these empty names do not denote objects in the world. Semantics, truth conditions, and ontology do not go hand in hand when we take into account that we use empty names as linguistic resources to reify. Having sketched an outline of this triadic theory of meaning, we can start analyzing the semantics of numerical statements.

As we have seen, numbers are mere discursive objects (or so I claim), missing worldly counterparts. Therefore, numerals must be empty names. In the terms of the triadic approach to semantics presented here, then, numerals refer to numbers (discursive objects) but do not denote any external objects. Notice that the distinction between reference and denotation is not one that users of numerical discourse are aware of. It is a theoretical distinction. Thus, for those operating in working-realist mode, referencing numbers amounts to speaking of numbers as if they were existing objects or, as I could say here, as if the mention of numbers spanned the two segments of the semantic relationship (reference and denotation). It is only an a posteriori scientific investigation that reveals that, in fact, the semantic relationship between numerals and numbers stops at the intermediate level.

This way of understanding the semantics of numerals reconciles two key aspects of numerical discourse. On the one hand, people do speak of numbers using syntactic resources typical of referential contexts where denotation is intended. On the other hand, the objects that apparently make up the domain of these discursive practices are not found anywhere other than at the discursive level. The adoption of a triadic semantics enables us to account for the referentiality of numerical discourse entirely within the mental/discursive level, without needing to postulate the existence of numbers at the worldly level.

This does not mean, however, that discourse about numbers is totally confined to the mental realm, without any link to the external world. As we saw above, reference in the first segment of the semantic relationship does not necessarily require denotation in the second segment for a relationship with the world to be established. As products of reification by the encapsulation of procedures, we can expect that the relationship between numbers and the world translates into events and practices. In fact, we have seen that numbers are cognitive tools which allow for the performance of certain operations. We can think of number concepts (of which numbers are reifications) as "scripts" to perform operations such as counting and calculating. This is clearly illustrated by the fact that numerical knowledge is tested by asking people to do something. The Give-a-Number task, which developmental

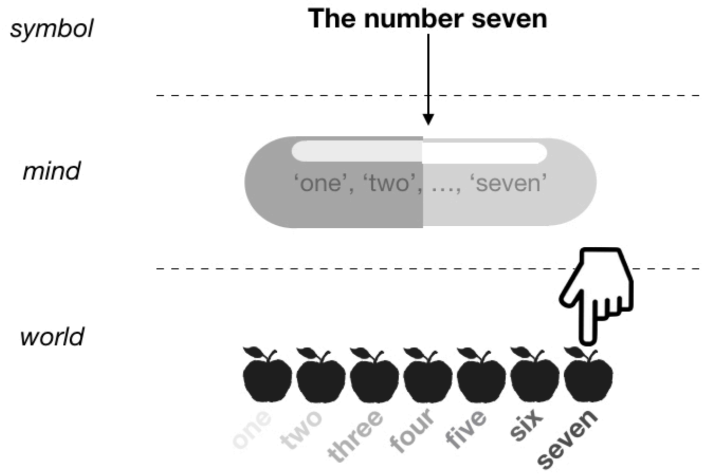

Figure 7.4: The discursive object seven, which encapsulates the procedure of counting up to 'seven,' does not denote anything in the world, but allows for the action of counting apples.

psychologists use to probe children's numerical understanding, is a case in point. According to this experimental task, a young child shows at least initial numerical understanding when she is able to produce a collection with the requested number of items. To use a rough analogy, we can think of the content of a number concept at earlier stages as a cake recipe. A cake recipe does not stand for a cake; rather, it instructs how to make a cake. Analogously, a number concept does not stand for nor represent an existing object; rather, it instructs how to perform certain operations. This is the relationship between numbers (discursive objects/mental contents) and the world: they allow for the performance of certain activities in the world, just like a cake recipe allows for the preparation of a cake. Figure 7.4 illustrates this in the triadic semantic schema. How this helps explain the truth conditions of numerical statements we will see in the next section.

7.2 Truth

As we saw above, the truth conditions of sentences wherein names of discursive objects produced by reification occur are specified by the truth conditions of their un-reified versions. To use Crane's terminology, the truth conditions of reified sentences are *reduced* to the truth conditions of their un-reified versions. The un-reified version usually reveals the relevant links between the original statement and the external world, and then we can evaluate the truth of the un-reified version following the standard notion of truth as correspondence with reality. To use Azzouni's terminology, we can say that these links with reality made explicit by the un-reified version are the *truth-value inducers* of the original, reified version. In this section, we will see how this approach explains the truth of applied and pure arithmetical statements.

Applied numerical statements Consider the following sentence: (a) There are seven marbles in the box. Its un-reified version un-capsulates and unalienates the counting event behind it, and can be phrased as follows: (a') If someone counts the marbles in the box, she will stop at the word 'seven.' The sentence (a') has no empty singular terms and its truth conditions are clear and unproblematic. It is easy to see that (a) is true if and only if (a') is true: (a') will be false only if the correct execution of the process of counting the marbles in the box does not stop at 'seven,' in which case there will not be seven marbles in the box, and then (a) will be false.

In (a), the numeral 'seven' is in adjectival position. Placing 'seven' in nominal position—as in (b): The number of marbles in the box is seven—does not change anything with respect to the truth conditions of (b), which are still given by (a'). However, this modification does change the semantics of (b) at the level of discourse, since now there are two expressions in (b) referring to discursive objects, whose identity the sentence asserts, according to the standard platonist analysis of the logical syntax of sentences like (b). If one wishes to preserve this semantical difference in the un-reified version of (b), an instance of Hume's Principle does the job: (b) The number of marbles in the box is seven iff (b') there is one-to-one correspondence between the marbles in the box and the numerals from 'one' to 'seven.' The right-hand side of this instance of Hume's Principle, (b'), gives the truth conditions of (b) in purely procedural terms. (The truth of Hume's Principle itself is discussed below.)

Both (a') and (b') show that the truth conditions of (a) and (b) depend on the cardinality of the collection of marbles in the box, as expected, since these sentences are asserting a worldly fact. The difference between reified and un-reified versions is that, whereas (a) and (b) just assert the cardinality of the collection of marbles in the box, (a') and (b') not only assert this but also make explicit the process we use to assess its cardinality. The specification of the truth conditions of applied numerical statements in terms of their un-reified versions is in line with correspondence-oriented and disquotational approaches. Here, the instance of the T-schema according to which

> 'The number of marbles in the box is seven' is true iff the number of marbles in the box is seven

is still valid. What is not revealed by mere disquotation, though, is what is behind the word 'seven' and, therefore, the links between applied numbers and the world remain obscure. By substituting (a') or (b') for the right-hand side of the above instance of the T-schema, these links are made explicit. Not only disquotation, but also un-reification is needed to account for numerals conceived of as empty names of discursive objects.

In section 1.5, I criticized Bueno's (2009) agnostic fictionalisms on the basis that his accounts require some rephrasing of arithmetical statements so that their epistemic content can be made visible. For example, although we do not know, in Bueno's accounts, whether (c) "There are infinitely many prime numbers" is true or false, we do know that (c') "If there were numbers, there would be infinitely many prime numbers" (in the van Fraassen-inspired account) or (c") "In arithmetic, there are infinitely many prime numbers" (in the Thomasson/Azzouni-inspired account). Are the changes from (a) to (a') and from (b) to (b') not also a kind of rephrasing? The answer is no. (a') and (b') do not rephrase the original sentences, but only specify their truth conditions, as the parallel with the T-schema shows. The sentences (a) and (b) are true (or false) as they stand. The point is that their truth conditions (their relationships with the world) do not directly reflect their syntactic form, given the occurrence of empty names that refer to discursive objects. In other words: in the account proposed here, the syntax and semantics of numerical statements is preserved, but their truth conditions are modified. The dissociation between syntax, semantics, and truth conditions is made possible by the inclusion of the intermediate semantic level of discursive objects (or intentional objects, in Crane's terminology), as explained in the previous section. These remarks apply to the reductive explanations of the truth of pure arithmetical statements we will see next.

Non-applied arithmetical equations Take the equation (d) 3+4=7 abstractly, i.e., as stating a fact about numbers, and not a fact about collections in the world. In this case, each numeral in (d) encapsulates a segment of the counting procedure abstractly conceived of, i.e., the action of simply reciting number words in sequence, without counting a particular collection. To see what the un-reified version of (d) is, recall young children's strategy "count all," through which we learn that 3+4=7, as we saw in section 6.2. To calculate 3+4, a young child would first deploy a collection of three items (counting 'one,' 'two,' 'three'), second deploy a collection of four items (counting 'one,' 'two,' 'three,' 'four'), and then count all the elements in both collections. In doing so, the child "unpacks" the numerals '3' and '4' and counts their "contents," as it were,[3] to find out what their sum is. The child needs to rely on collections of physical items because she has not yet reified number concepts, as we saw in section 6.3. But it does not matter whether she uses fingers, beans, or pebbles to implement this operation. What she is learning is the following procedure:

> (d') if someone counts 'one,' 'two,' 'three,' and then counts 'one,' 'two,' 'three,' 'four,' and then counts what she has counted in the previous steps together, she will stop at 'seven.'

This is a fully un-reified and unalienated version of (d). Empty names do not occur in (d') and it is easy to see that (d) is true iff (d') is true. Notice that the truth of (d'), as opposed to the truth of the examples of applied numerical statements considered above, does not depend on the cardinality of any particular collection in the world. The sentence (d') states a fact about the counting procedure itself. It is a condensed form of describing what happens if someone counts three, counts four, and then counts all. In the absence of objects to be counted, one can count the very number words to verify it.

A perhaps more elegant way of saying what 3+4=7 describes about the counting procedure comes from the "count on" strategy also used by children to calculate additions. In this strategy, the child starts at 'three' and counts "four, five, six, seven." That is, the child moves from 'three' to four positions ahead in the counting sequence. The equation '3+4=7' can be seen as encapsulating this action. In other words, 3+4=7 is true iff moving from 'three' to four positions ahead in the counting sequence results in stopping at 'seven.' This phrasing, in contrast to (d'), is still reificatory, since it is referring to four. For a less reified graphic

[3] More precisely, the child counts together the collections whose items can be put into one-to-one correspondence with the counting words recited during the execution of the initial segment of the counting procedure encapsulated by each numeral.

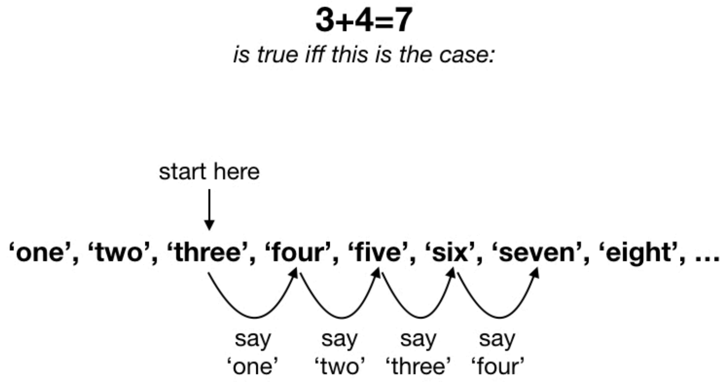

Figure 7.5: 3+4=7 is true iff moving from 'three' to four positions ahead in the counting sequence results in stopping at 'seven.'

description of the process encapsulated in 3+4=7 inspired by the count on strategy, see Figure 7.5. Using a still more reified way of speaking, it can be said that '3+4=7' describes a structural property of the counting sequence according to which the fourth numeral after 'three' is 'seven.'

Recall that, as we saw in Chapter 6, we learn arithmetical operations as higher-order operations, i.e., operations performed with the counting operation. The second-order nature of addition is clearly depicted in Figure 7.5. In general terms, an equation of the form $a+b=c$ describes the outcome of the action of moving b positions forward in the sequence of counting words from position a. Similar analyses can account for the truth of subtractions, multiplications, and divisions of natural numbers. Subtraction equations describe actions of moving backward in the sequence of numerals; multiplication equations describe actions of leaping forward a number of times in the sequence of numerals; and divisions, actions of leaping backward a number of times. See Figure 7.6 for an illustration.

Arithmetical equations describe the functioning of the cognitive tool of counting, and become themselves cognitive tools for calculation. Once we have learned facts about the counting sequence, we can make use of them to save labor. For example, if we already know that a basket contains three apples and four oranges, we do not need to count all the items in the basket to learn that it contains seven fruits, since we already know that, if we counted them all, we would stop at 'seven.' We can also make use of facts about the counting procedure for making predictions. We do not need to add another three oranges to the basket and count all the oranges to know that this would result in seven oranges in the basket. Once we know these properties of the counting procedure, i.e., once we know what happens if we count this way or that, on certain occasions, we no longer need to count, since we can just calculate (e.g., by retrieving the outcome from memory). But notice: arithmetical equations are not generalizations of facts about collections. They describe structural properties of the counting procedure, regardless of any particular collection (other than the very ordered collection of counting words).

To calculate is to use previously known facts about the counting procedure so that one no longer needs to reiterate the counting procedure every time (as young children do). Cal-

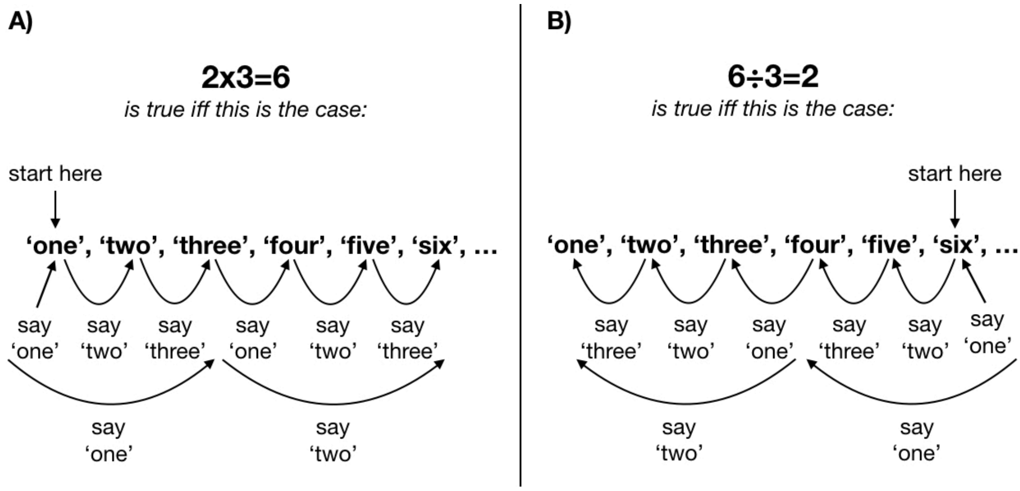

Figure 7.6: (A) illustrates multiplication as a third-order counting operation. The product of 2×3 is the position in the counting sequence one arrives at after counting up to three twice. (B) illustrates division as a third-order counting operation. The division of 6 by 3 is the number of times that it is possible to count up to three backwards in the counting sequence, starting at 'six.'

culation algorithms based on Arabic digits increase our calculation capacity even more by providing "shortcuts" which dispense us from memorizing a potentially infinite list of facts about the counting procedure. We just have to memorize the facts compiled in addition and multiplication tables and know the algorithms. The algorithms take us directly to the outcome that would be produced if we moved through the counting sequence in the way specified by the intended operation. It is much easier to find out the product of 748 times 465 by using the multiplication algorithm than by counting up to '465' 748 times over the counting sequence to find out which numeral this operation will stop at. The multiplication algorithm encodes basic facts that are established by counting (such as that 8×5=40) and exploits features of the notation system to reduce the number of counting facts we have to memorize.

As we saw in section 2.4, calculation algorithms are cognitive tools in their own right. "The decimal place-value system is both a *medium* for representing numbers and a *tool* for operating with numbers. This invention creates what can be termed a 'symbolic machine'" (Krämer, 2003, p. 531-2). Now we can appreciate the fact that this "symbolic machine" is a second-order cognitive tool, i.e., a cognitive tool that operates using another cognitive tool, namely, counting.

As we also saw in section 2.4, the process of de-semantification that takes place when we use calculation algorithms makes reference to numbers irrelevant. In a sense, by calculating mechanically, it is as if we go back to the asemantic origins of counting words: a mechanical, rule-governed technique does all the work, and meaning is just a product of the mechanical manipulations, rather than their cause. This highlights the *cognitive* role of numbers as as-if objects: they serve to endow such operations with meaning *for us*, to make *our* understanding and *our* execution of these operations easier, but they are irrelevant for the operations themselves. This is why machines, which do not understand meanings, can count and calculate.

This is why procedures, rather than numbers, ground arithmetical knowledge: procedures are essential, whereas numbers are dispensable (for machines, at least; for us, they are cognitively indispensable for facilitating complex operations, as we saw in Chapter 6).

A few paragraphs above I said, using a reified way of speaking, that arithmetical equations describe "structural properties of the counting sequence" or "the functioning of the cognitive tool of counting." However, we need to be very careful with this way of speaking. On the one hand, it conveys easily and clearly the point I am trying to make here, as usual with reificatory ways of speaking. On the other hand, these reified expressions can obscure the connections non-applied arithmetical equations have with the world, which I also want to clarify here. There is a tendency to understand expressions like 'the process of *x*' as having to denote either a specific execution of the process or a non-spatiotemporal entity, like a "procedure type" which is instantiated in each execution. These interpretations should be avoided here. "The process of counting," "the counting sequence," and "the cognitive tool of counting" do not exist in themselves; what does exist are human beings who know how to execute a ritualized sequence of actions which, time and again, they do.[4] These human beings have mental contents that allow them to execute these actions when required. These mental contents exist, of course, but they are not "the counting procedure" either, nor are they the truth-value inducers of arithmetical equations. These mental contents, as said at the end of the previous section, can be conceived of as scripts, or "cake recipes" that instruct the agent how to act in order to count or calculate. Arithmetical equations do not describe these mental scripts, but what would be the outcome of the execution of these scripts. This is the relationship between arithmetical equations and the world: they describe what happens if we proceed in a certain way, just like a picture of a cake anticipates the outcome of following a cake recipe.

Arithmetical statements ranging over infinity As we saw in section 6.4, statements about actual infinity encapsulate unending processes. For example, the sentence (e) "The set of the natural numbers is infinite" encapsulates the procedure of successively adding one, or any other procedure that generates increasingly large numbers. Therefore, a semi-un-capsulated and semi-unalienated version of (e) may be the following: (e') If someone starts adding one to a number repeatedly, there is nothing in this process itself that will cause her to stop, and hence she can keep adding as long as she can or wishes. It is easy to see that (e) is true iff (e') is true. Notice that (e') is true in spite of no one being able to keep adding forever. This does

[4]This kind of approach to procedures is not strange in logical and mathematical contexts. For example, what we today call a Turing machine and see as an abstract computational device was originally introduced by Turing (2004) as a procedure executed by a human agent. Interestingly, the features of this "machine" were motivated by "Turing's recognition of a set of human limitations which restrict what we can compute (of our sensory apparatus but also of our mental apparatus)" (De Mol, 2019, section 3.1, para. 2). For example, the fact that the head can read/write only one cell at a time is motivated by Turing's recognition that there is a limit to the number of symbols we can observe at one glance: "If he [the human computer] wishes to observe more, he must use successive observations" (Turing, 2004, p. 75). It is not difficult to see that the limitations that prevent a human computer from reading several cells at one glance may be related to the cognitive constraints that prevent us from grasping the cardinality of collections with more than three or four items at one glance, forcing us to rely on a procedure—counting—where we use successive observations. Similar technical problems lead to similar solutions.

not matter for the truth of (e') because (e') states a fact about the procedure of adding one, not about specific executions of it. The procedure of adding one has no intrinsic limit; it can go on forever. We know that the procedure of adding one has no intrinsic limit not merely by executing it—although by executing it we can convince ourselves that this procedure is potentially unending—but also because it is governed by clear-cut, deterministic rules, which allow the use of deductive reasoning to discover facts about it. By a simple *reductio*, we can deductively conclude that it never ends.[5]

Other statements ranging over infinity have different truth-value inducers, according to the procedure they encapsulate. Take, for example, the sentence (c): There are infinitely many prime numbers. There is no method for generating ever-larger prime numbers, but there are methods for testing whether or not a number is prime. Thus, (c) can be seen as a reificatory manner of expressing the observation that, if one keeps testing numbers, she will always find more primes. As above, the truth of this observation is not based on the experience of continuously testing and always finding new primes (and here, continuous testing could not even convince us of this fact). What informs us that (c) is true is a deductive proof of (c), such as Euclid's proof. The underlying facts that make (c) true, however, are still the rules that determine the sequence of numerals and the operation of division. The proof only reveals these facts. This illustrates how the truth-value inducers of a sentence (in arithmetic as in other disciplines) can be different from the methods we use to determine its truth value. Again, using reificatory language, it can be said that (c) captures a structural property of the operation of division performed over the counting sequence.

Hume's Principle Neo-Fregeans claim that Hume's Principle is analytic, i.e., true in virtue of the meaning of its constituent terms. If we take the un-capsulation of its constituent terms as revealing aspects of their meanings, the analyticity of Hume's Principle seems to be confirmed. Consider the following phrasing of Hume's Principle: (a) The number of Fs is equal to the number of Gs iff F is equinumerous to G. On the left-hand side of (a), the reificatory expression "the number of …" occurs twice. Only this side needs to be un-reified, since equinumerosity is already defined by means of the process of establishing one-to-one correspondences. The following is a way of un-reifying the left-hand side: if someone counts Fs and counts Gs (separately), she will stop at the same numeral. Now, recall that counting is establishing one-to-one correspondence between the elements of the collection being counted and an initial segment of the counting sequence. Therefore, what the left-hand side says is that F and G are equinumerous to the same initial segment of the counting sequence. By the transitivity of equinumerosity, this means that F is equinumerous to G, and this is exactly what the right-hand side of Hume's Principle states. Therefore, Hume's Principle cannot be false.

[5] Suppose that the process of adding one has an end. Then, at its end, we get the number n, the last one to which we can add one. However, by making n+1, it is possible to go on from n, contradicting the assumption. Notice that here and in (e') I am referring to the number one. (e') is not a fully un-reified version of (e), but a fully un-reified version can be obtained by un-capsulating the operation of adding one in the way described above—it is the operation of moving one position ahead in the counting sequence. As we know from section 6.4, the counting sequence is finite, and therefore there comes a point where it is impossible to go on. This is a limitation of the medium, not of the procedure. I address this issue in the discussion about Peano's Axioms below.

Peano's Axioms Consider the following formulations of Peano's Axioms:

(1) 1 is a natural number

(2) For every natural number n, $s(n)$ is a natural number

(3) For every natural number n, $s(n) \neq 1$ (1 is not the successor of any number)

(4) For all natural numbers m and n, if $s(m)=s(n)$, then $m=n$ (for every pair of numbers, their successors are different from each other)

(5) If Px is any property such that (i) $P1$ and (ii) for every natural number n, if Pn then $Ps(n)$, then Px for every x natural number.

These axioms give the standard definition of natural number. As a nonalgebraic theory (in Shapiro's (1997) terminology), Peano Arithmetic is intended to be true about a definite subject matter. In the working realist view, it is true about the actually infinite set of natural numbers. In the current account, this means that here we have two levels of encapsulation. At the first level, each number encapsulates a subprocess of the counting procedure. Since here numbers are being taken abstractly, i.e., not as referring to cardinalities of particular collections, the procedure encapsulated into each number n is just the action of reciting number words in ascending order up to 'n.' This gives us a potentially infinite sequence of natural numbers, since the action of reciting number words is potentially unending. At the second level of encapsulation, the very procedure of obtaining ever larger numbers is encapsulated into 'the set of natural numbers' or $\mathbb{N}$. Peano's Axioms describe this discursive object, $\mathbb{N}$.

By undoing the two levels of reification that give rise to $\mathbb{N}$, we end up with the counting procedure abstractly conceived, i.e., the action of reciting counting words without counting any particular collection. This suggests that, ultimately, Peano's Axioms, just like arithmetical equations, describe properties of the sequence of counting words. However, as opposed to arithmetical equations, which describe specific facts about the sequence of counting words, Peano's Axioms capture essential properties of fully-fledged counting sequences.[6] From this perspective, the first four axioms use reificatory language to state the fact that the process of reciting a counting sequence is potentially unending and that it does not make loops. The following are semi-un-reified versions (and fully un-reified and unalienated versions in brackets) of the first four axioms:

(1') the process of reciting a counting sequence starts with saying the numeral 'one' (if someone is to recite a counting sequence, she will start by saying 'one,' or the corresponding numeral in the sequence she is using)

[6] A fully-fledged counting sequence is one that has numerals for higher numbers, such as the English sequence, in contrast to counting sequences whose upper limit is low, such as those used in some small-scale societies. As we saw in section 6.4, findings from the fields of numerical cognition and developmental psychology suggest that experience with higher numbers is required for the development of the idea of an infinite series of numbers, and only fully-fledged counting sequences can provide this kind of experience.

(2') once a numeral '*n*' is recited, the action proceeds by saying the numeral that is the successor of '*n*' (if someone is reciting a counting sequence, after reciting a numeral '*n*,' there is nothing in this process itself that can prevent her from proceeding by reciting the numeral that is the successor of '*n*' as long as she can or wishes)

(3') 'one' is never recited as the successor of any numeral (if someone is reciting a counting sequence, she never says 'one' or the corresponding numeral in the sequence she is using as the successor of any numeral)

(4') one and the same numeral is never repeated as the successor of different numerals (if someone is reciting a counting sequence, she never repeats one and the same numeral as the successor of different numerals)

Notice that when we step back from actual infinity to potential infinity, temporal relationships need to be reintroduced. This is why words such as 'start,' 'proceed,' and 'never' occur in the un-reified versions of the axioms. It is also worth noticing that the statements (1')-(4') describe a *correct* recitation of a counting sequence from its beginning. Surely, someone can start at a numeral other than 'one,' or can say 'one' as the successor of some numeral, or repeat one and the same numeral several times. But all of these variations will count as incorrect recitations (I discuss the normativity of the counting procedure in section 7.3).

Another point that requires clarification is related to (2'). We know from section 6.4 that in all languages, counting sequences end. This is one of the reasons why we cannot speak of infinity without reifying. But, just like other physical constraints that prevent us from counting forever, the fact that we will eventually run out of numerals is not an intrinsic limitation of the process of reciting a counting sequence. This process could go on if additional numerals were provided. This is a limitation of the medium—the particular counting sequence used—wherein the process is implemented. This is shown by the fact that, varying the medium, the limit at which one runs out of number words also varies. For example, the limit in Komnzo, the language spoken by the Farem people in New Guinea, is the numeral for 6^6 (Döhler, 2018), much smaller than the limit in English, but the process implemented in both is the same. Furthermore, ultimately, one can go on by writing down numerals in Arabic notation, which is truly potentially infinite, or by inventing new number words, or by using arithmetical operations (saying "'*n*' plus one," "'*n*' plus one plus one," and so on).

Ferreirós (2016) gives a rather similar account of the truth of the first four of Peano's axioms in virtue of the counting sequence. He claims that "the [Peano] axioms are true of the counting numbers, that is, by reflection on the practice of counting we can realize that the axioms are correct, true statements" (Ferreirós, 2016, p. 190), as do I. However, he conflates numbers and numerals—"the counting numbers are simply the numerals employed for counting, or any symbols employed to represent them ..., as determined by the rules for their generation" (Ferreirós, 2016, p. 189)—and then takes the axioms to be about the rules for the generation of counting words:

> The first three Peano axioms are rather trivial; they merely capture basic features of the process of recursive generation of number-words; as a matter of fact, a precise statement of the generative rules we employ for numbers in any vernacular language is much more complicated than the above set of axioms. The fourth axiom just says that

> whenever two numbers are different, $m \neq n$, their successors are also different—and this is also a simple fact based on the generative rules (Ferreirós, 2016, p. 191).

The problem with Ferreirós's account is that the first four axioms do not capture features of the process of the recursive generation of number words. As mentioned above, recursively generated number words are just a medium with which we commonly execute the process of reciting counting words, but the axioms are true of the process itself, regardless of the medium. For example, the axioms would still be true if we used a sequence of randomly generated tokens as a medium, not following any recursive rules. But Ferreirós's view is understandable, since we learn that the process of reciting a sequence of numerals is intrinsically unending based on the recursive rules for the formation of number words, as we saw in section 6.4. Historically and ontogenetically, the idea of numbers constituting an infinite set could hardly have emerged if a recursive system of notation was not available. This particularity, though, is irrelevant for the truth of the axioms. When we reify number concepts, we no longer see them as attached to number words; numbers are abstracted from the words that gave rise to them, and in this process, particularities of the medium become irrelevant. The axioms capture structural properties of the sequence of numbers, as a working realist could say. Or, as I would say, the truth-value inducers of Peano's Axioms are the essential properties of fully-fledged sequences of numerals. These essential properties are structural: they pertain to the *positions* numerals occupy in such sequences, regardless of the intrinsic properties of the occupants of these positions.[7]

The axiom of induction, the fifth axiom, demands a different story. The role of this axiom is to allow proofs by induction in arithmetic. This axiom does not describe an essential property of the process of reciting a counting sequence, as the others, but another operation. In procedural terms, what the axiom of induction states is that one cannot obtain a proper subset of the natural numbers that contains the number one and is closed under the successor function (Enderton, 1977, p. 71). If one tries to do this, she will end up with the entire set of numbers. This procedural formulation of the axiom of induction suggests the following un-reified and unalienated version of (5):

> (5') if someone is to recite only numerals that satisfy a certain structural property, and 'one' (or the corresponding numeral in the sequence she is using) satisfies this property, and for any numeral that satisfies this property its successor also does, then she will perform the whole procedure described in (1')-(4').

As above, (5') assumes a *correct* assessment of the relevant property. One could object that numerals and numbers have different properties, and therefore (5') could at most be true of properties of numerals. But recall that, when statements about numbers are un-reified, they become statements about processes performed with numerals, and therefore properties of numbers are converted into properties of numerals. For example, the commutative property, according to which $m+n=n+m$, here has its truth reduced to the observation that the operation of moving from 'm' to n positions ahead in a counting sequence stops at the same position as

[7] This resembles Shapiro's (1997) and Resnik's (1997) structuralism. However, keep in mind that, in the current account, "the natural-number structure" is just a reified way of speaking. There is no "*ante rem*" structure described by Peano's Axioms, and numbers are not positions in a structure.

moving from 'n' to m positions ahead. In this way, properties of m and n are translated into properties of 'm' and 'n.'

Before moving on, let me briefly address a general objection that can be leveled against my account of the truth of Peano's Axioms. One may call attention to the fact that Peano's Axioms are *axioms*. As such, they are supposed to be the basic truths which ground all of arithmetic—including the counting procedure and the sequence of numbers words. Thus, or so the objector may claim, what I am doing here is mistaking the *explicandum* for the *explicans*: it is the sequence of counting words that is true in virtue of the axioms, not the other way around. As a response to this objection, I offer the argument I have been developing since Chapter 2: in ontogenetic and historical terms, meaningless symbols used for tallying or counting precede number concepts, which suggests that, in reality, it is these practices, rather than highly abstract axioms which are a late development in the history of mathematics, that ground arithmetical knowledge. We have good reasons to think that counting is a technique created to help us cope with large cardinalities accurately, and that pure arithmetic is a discipline which investigates and further develops this technique. What I showed in this section is that this reversal does not change anything with respect to the truth of arithmetic. Peano's Axioms are a concise way of characterizing the most basic structural properties of the cognitive technology of counting.

7.3 Normativity, necessity, apriority, objectivity

According to platonism, arithmetic is descriptive (it describes how things are in the non-spatiotemporal realm of numbers), objective (for the same reason), a priori (numerical knowledge does not come from experience) and necessary (facts about numbers are independent of how the world could be). In this section, I will show why arithmetic as construed here is also descriptive, objective, a priori and can be thought of as necessary, though not in the same senses these attributes are conceived of by a platonist.

Normativity Let me start, as usual, with the counting sequence. In platonist accounts, the counting sequence is seen as representing the sequence of numbers in ascending order, and the normativity of counting is derived from its primarily descriptive nature. As Frege puts it, "[a]ny law asserting what is, can be conceived as prescribing that one ought to think in conformity with it, and is thus in that sense a law of thought" (Frege, 1964, p. xv). Applied to platonist arithmetic, this means that insofar as the counting sequence represents what is the case with respect to numbers, it has normative force in our numerical thoughts and practices. The idea is that the rules for correct counting are as they are because numbers are as they are.

In the current account, numbers do not exist, and thus the counting sequence cannot represent them. Here, the counting sequence is just a model collection used in a specific kind of tallying technique where an ordered set of words replaces physical objects as the items to be matched with the items of target collections. The use of these words is regulated by five principles—the counting principles (Gelman & Gallistel, 1978) presented in section 4.3—which are exclusively prescriptive: they say what one has to do if she is to accurately determine the cardinality of a collection by counting it. The counting principles solely prescribe, without describing anything. Their normative force does not come from their being true descriptions of an underlying reality—they are not even true or false, since they are prescriptions—but from their being enforced by *hypothetical imperatives*.

Hypothetical imperatives were originally introduced by Kant, in contradistinction to categorical imperatives. Hypothetical imperatives express commands that have normative force for an agent conditional on what she wants, whereas categorical imperatives express commands that have normative force for everyone independently of their aims (Kant, 1998, 4:414). The relevant point here is that hypothetical imperatives are if-then clauses in which the antecedent states a goal and the consequent states a means to achieve the goal. For example, "if you want to be healthy, then you have to exercise." The link between antecedent and consequent and the resulting normative force of the conditional arises from practical considerations. Which means are suitable to achieve health is not determined by conventions, laws of logic, physics or metaphysics. Rather, means to achieve health are determined by the capabilities and characteristics of the agent who has this goal and the relevant environmental conditions. For this reason, Kant also calls hypothetical imperatives "rules of skill" or "technical imperatives" (Kant, 1998, 4:415-16). Whether the agent will feel compelled to adopt the means determined by technical imperatives or not, depends on her really intending the goal, knowing which are the possible means to achieve it, and displaying instrumental rationality, i.e., the disposition to adopt suitable means to her ends.

The counting principles are enforced by a number of hypothetical imperatives. To make this point clearer, let me briefly recall the five counting principles: (1) one-to-one correspondence (each item of the target collection must be paired with one and only one number word); (2) stable order of counting words (the order in which number words are used must be consistent across different counting events); (3) order irrelevance of items (the order in which items of the target collection are counted may vary across different counting events); (4) abstraction (counting applies to any collection of all sorts of objects); and (5) cardinality (the final number word used in a counting event represents the cardinality of the whole collection).

The first principle, one-to-one correspondence, is mandatory for counting due to our cognitive limitations. As we have seen in Chapter 3, our quantical abilities do not deliver accurate estimations of the cardinality of collections with more than three or four items. Given this constraint on our quantical abilities, the following hypothetical is imperative for us: if one wants to determine the cardinality of a collection with more than three or four items accurately, then she has to match it with a model collection. The antecedent of this hypothetical is the primary goal of counting. But this is not its only goal. As we saw in Chapter 5, there are simpler tallying techniques that do not implement all the counting principles, but even so allow for the assessment of larger cardinalities accurately. This is possible because the remaining counting principles derive normative force from other goals not related to accuracy.

The second counting principle, stable order of counting words, is imperative if one wants to benefit from the fifth principle, cardinality. The counting word used for tagging the last item in a count can represent the cardinality of the whole collection only if counting words are always used in a stable order. The cardinality principle, in turn, derives its normative force from the goal of being concise and expeditious: someone who does not want to be concise and expeditious can simply recite all the counting words up to the last one used in a count every time she is required to discern the cardinality of a collection. Recall that this is what young children do before they have learned the cardinality principle, as we saw in section 4.3: when asked "how many" just after they have finished counting a collection, they count it again (i.e., they recite all the number words again). Tallying techniques implemented with sticks or other physical items rely on a similar *modus operandi*: in order to refer to the cardinality of a collection, the whole model collection has to be displayed. By using the cardinality principle, one produces the same effect by "displaying" (saying) only the last item of the model collection.

The third counting principle, order irrelevance, derives its normative force from the goal of being efficient: it is counterproductive and futile to keep the order in which items are counted constant across different counting events. Finally, the fourth principle, abstraction, is imperative for the sake of efficiency and universality: it is counterproductive and unnecessarily specific to have different counting procedures for different kinds of items. Notice that it is entirely possible to comply with cardinality, stable order and one-to-one correspondence without complying with order irrelevance and abstraction. This is possible because the latter derive their normative force from different goals than the former.

Now we are in a position to appreciate why the sequence of counting words does not make loops, as captured by the un-reified versions of Peano's Axioms. If the sequence of counting words made loops, or if one and the same numeral could be the successor of different numerals, the counting word used for tagging the last item could not represent the cardinality of the counted collection, thus violating the cardinality principle (precluding conciseness and expeditiousness). More fundamentally, this would also violate one-to-one correspondence, since after the first loop, different items of the target collection would be associated to the same counting word (precluding accuracy). Notice that the counting principles do not imply infinity; they are compatible with the finite sequences of numerals available in natural languages, which are suitable for everyday purposes. Infinity becomes mandatory only in virtue of the goals of full generality and unboundedness typical of scientific and mathematical contexts.

That being the case, arithmetic is prescriptive at its roots in the counting procedure. But arithmetic as construed here also has a descriptive dimension. Pure arithmetic or number theory (the branch of mathematics that studies the properties and relationships of the natural numbers) is descriptive in that it is an investigation of the functioning of counting and calculation techniques, or, in other words, it is descriptive of the conceptual structures that emerge from these practices through reification. For example, Euclid's Theorem about prime numbers describes a structural property of the sequence of numbers and the operation of division; arithmetical equations describe structural properties of the sequence of numerals and the operations of addition, subtraction, multiplication and division; and Peano's axioms describe the most basic structural properties of counting sequences. These descriptive statements also generate prescriptions through the mechanism pointed out by Frege, according to which any true description is also a prescription, in the sense that we have to think and behave in accordance with it. For example, the algorithms for calculating additions, subtractions, multiplications and divisions with Arabic digits make use of arithmetical facts compiled in addition and multiplication tables to prescribe more efficient calculation techniques.

In its prescriptive dimension, arithmetic is a toolkit of techniques whose primary purpose is to enable us to deal with cardinalities accurately and efficiently; in its descriptive dimension, arithmetic describes structural aspects of these cognitive tools.

Necessity The observation that the prescriptions made by arithmetic are given their normative force due to hypothetical imperatives sheds light on the senses in which arithmetic can be said to be necessary or contingent. It goes without saying that people who do not have the relevant goals are not bound by the technical imperatives that enforce arithmetical procedures. For example, in certain cultures, there are taboos against counting certain kinds of things, or different counting sequences are used depending on what is counted (Zaslavsky, 1999). People in these cultures do not value universality with regard to counting as we do, and therefore are not bound by the principle of abstraction. More dramatically, in some cultures, there is no need to determine the cardinality of collections with more than three or four elements accurately. The Pirahã are a case in point. As we saw in Chapter 5, they do not count, nor use any other tallying technique. They are not bound by any of the counting principles. As a rule, the normativity of arithmetic is *contingent* on people's goals.

This means that the *emergence* of arithmetic is contingent. Arithmetic will not emerge in societies where people do not have the relevant goals. This, however, does not imply that arithmetic itself is contingent. It is remarkable that such dissimilar human cultures have developed such similar tallying, counting, and calculation techniques, often in isolation from each other. This is because the technical prescriptions that recommend one-to-one correspondence (and the other counting principles) are *imperatives* for us. Those who do not have the relevant goals are not bound by them, but for those who do, these technical prescriptions are *necessary*. Recall the discussion about the internalization of cognitive tools in section 2.3. Culturally created cognitive tools must find their neuronal niche; that is, a cognitive tool will not be created (or will not be usable, efficient or efficacious) if there are no previously existing brain functions and capacities that it can exploit and recycle/reuse. Neuronal recycling is an endogenous force that shapes cultural creations. Given that we, as humans, share the same genetic endowment, it is just natural that we are bound by the same cognitive constraints. For example, one-to-one correspondence is unavoidable for those of us who want to determine larger cardinalities accurately because we cannot keep track of long chains of subitizing without a memory aid (a model collection). This means that if the Pirahã or any other anumeric people ever develop a technique to assess cardinalities precisely, they will develop a tallying or counting technique following at least the one-to-one correspondence principle and possibly some of the other principles, if they value conciseness, expeditiousness, efficiency, and universality. Since all typical humans are subject to the same technical imperatives, they will develop counting and calculation techniques with the same structural properties (i.e., the same arithmetic).[8]

When it comes to humans with the relevant goals living in this world, then, arithmetic is necessary (in the sense that it could not be otherwise). But is arithmetic necessary in

[8]Tallying, counting, and calculation techniques are remarkable examples of *cultural attractors* (Scott-Phillips, Blancke, & Heintz, 2018; Sperber, 1996). The observation that the counting principles derive their normative force from technical imperatives explains why they are cultural attractors.

general—true in all possible worlds? This question is as speculative as it gets. In the absence of information about all possible worlds, we can rely only on our *intuitions* about possibilities and impossibilities, forged in this world, to address it. This is not a reliable method (Machery, 2017). However, given the attention this question has received in the literature, I cannot avoid it here. I warn the reader that from here on, I proceed speculatively.

Intuitively, it is plausible that there are possible worlds where humans have not invented arithmetic, and possible worlds where humans do not exist, as well as possible worlds where beings capable of language do not exist. In platonist accounts, arithmetical statements can be true in all possible worlds because they express platonic propositions whose existence is independent of the existence of beings who can formulate them. In the current account, though, arithmetical propositions are human creations, and therefore they cannot be true in worlds where they are not asserted or thought of. In this sense, arithmetic, as construed here, is clearly contingent: there are worlds where arithmetical propositions do not exist, because no one has invented the practices that could give rise to such propositions, and therefore they are not true in these worlds.[9]

However, there is another sense in which we can think of arithmetic as construed here as necessary. Hale, in Hale and Wright (2001, p. 185), distinguishes between being *truly asserted in* a possible world and being *true of* a possible world. Arithmetical statements cannot be truly asserted in possible worlds where cognitive agents capable of asserting them do not exist, or where the practices these statements describe do not exist. However, from our perspective, in this world, where such practices do exist, we can assert arithmetical statements that can be true of those possible worlds. For example, we can assert (in our world) of a certain world w that, if someone from our world counts two objects in w, and counts another two objects in w, and counts all of them, she will stop at 'four.' This statement will be true of w insofar as 2+2=4 is true of episodes of addition in w. But recall that arithmetical statements, in this account, reflect structural properties of procedures, and not structural properties of the world. This suggests that 2+2=4 will be true of episodes of addition in all possible worlds.

To see this point, consider a bizarre possible world w_b where, whenever someone puts together four items, making up a collection, one of them disappears.[10] If this happens to all collections, three is the largest size a collection can have in w_b. People living in w_b will probably develop rigid taboos against collecting more than three valuable items so as to avoid losing one or more of them. Could we truly assert of episodes of addition in w_b that, there, 2+2=3? Absolutely not; if 2+2 were *equal* to 3, the inhabitants of w_b would not miss the lost item. What we can truly assert of w_b is that there, 2+2 *becomes* 3. It is still true of w_b

[9]This conclusion does not follow if propositions *in general* are taken to be platonic entities. Recall that, in the formulations presented in the previous section, the un-reified versions of arithmetical statements, in virtue of which they are true, are expressed in the form of conditionals. For example, the un-reified version of the first Peano's Axiom runs as follows: "if someone is to recite a counting sequence, she will recite 'one'". Thus, in possible worlds where counting sequences and calculating practices do not exist, arithmetical propositions are vacuously true. In the following paragraphs I show that the consequents of the un-reified versions of arithmetical statements cannot be false when the antecedents are true. Thus, assuming a platonist stance with regard to propositions, arithmetic as construed here can be seen as true in all possible worlds.

[10]This and the following examples echo Wittgenstein (1978, p. 51, § 37). However, the conclusion I draw from these examples diverges from Wittgenstein's.

that, there, 2+2 is equal to 4. What happens is that a modification in the environment takes place whenever four items are put together: the moment two is added to two, a collection of four items is formed and then one is subtracted. Our arithmetical statements are still true of counting episodes and calculations about objects in w_b. For people living in w_b, it may be undesirable to count beyond three, since every time one tries to do so, items start disappearing. However, it is still true of w_b that, for example, if one hundred items were to be collected, ninety-seven of them would disappear. If the inhabitants of w_b managed to develop a way of counting abstractly, as we do with numbers, they could realize this arithmetic fact by themselves.

Arithmetic is not invalidated even in starkly different worlds where there are no enduring, clear-cut objects. Imagine, for example, a homogeneous liquid world, where there are no discrete objects; or a world where ephemeral objects keep popping up and vanishing all the time, like rain drops floating in the air and fusing with each other or breaking apart continually. As I speculated in section 3.5.3, in such worlds, perception of numerosity is unlikely to emerge, let alone counting practices like ours. However, our arithmetic is still applicable there. A numerate human who traveled to the rain-drops world could take a snapshot of the ephemeral rain drops, and then count how many of them there are in the snapshot. In the liquid world, a numerate human traveler could divide the space into discrete units by means of an arbitrary system of three-dimensional coordinates, and then count how many of these units there are in a certain region. In fact, this is what we do when we measure continuous quantities (e.g., volume in cubic meters).

The observation that arithmetic is about procedures, rather than about the structure of our world, shows that arithmetic is not invalidated in possible worlds with different structures. But could the procedures that define arithmetic be otherwise? As I already mentioned, as far as human agents are concerned, arithmetic could not be otherwise, since it is enforced on us by technical imperatives.[11] However, we can imagine that in some possible world, there are more powerful cognitive agents who are not subject to the same technical imperatives as us. For example, they may be able to assess larger cardinalities without necessarily having to use a technique based on one-to-one correspondence. In this case, if they ever needed to develop cognitive tools for assessing cardinalities and calculating accurately, their techniques may be different from our numbers. Even so, insofar as their cognitive tools are accurate, they will be compatible with ours. What validates a method of assessing cardinalities is correspondence between the outcomes produced by the method and the cardinalities evaluated. Therefore, two accurate methods of assessing the cardinality of finite collections[12] are likely to agree in their outcomes.[13]

[11] Wittgenstein's pupil (Wittgenstein, 1958, § 185), for whom 1000+2 is 1004, 1004+2 is 1008, and so on, is not a counterexample. He can be accused of not following the technical imperatives the practice he is engaged in requires. Maybe everyone is free to reinterpret a rule as she pleases and still claim that she is following the same rule. But if the rule is part of a technique that is meant to achieve an end, there are *mis*interpretations of the rule that will not do.

[12] For infinite sets, different methods may produce different assessments of cardinality (Benci & Di Nasso, 2003; Mancosu, 2009).

[13] That is why Wittgenstein's pupil's additions above 1000 can be dismissed as simply wrong. What shows that they are wrong is not the conventional meaning of additions, but the end of the practice of calculating additions (to predict the resulting cardinality accurately), which is not met by the pupil's procedure.

All things considered, arithmetic is contingent in that it is a human creation; humans will not invent it if they are not compelled to do so by certain needs. On the other hand, once it has been invented and as far as typical humans are the inventors, the procedures that define arithmetic could not be otherwise, provided that all the relevant goals are present. Cognitive agents who are not subject to the same cognitive constraints as us may be able to develop a different arithmetic, but as far as the goal of accuracy is concerned, their arithmetic and ours will be compatible in their outcomes. Furthermore, correct arithmetical statements made in this world will likely be true of any possible world, and in this sense arithmetic is probably necessary (in the sense of true of every possible world). I say *probably* not only because the discussion of this point is speculative, but also because there may be other factors, which I did not take into account here, that can make arithmetic, or parts of it, false under certain circumstances. For example, it may be that arithmetic as we know it (i.e., number theory conceived of as a descriptive theory) is ultimately inconsistent. In this case, the arithmetic we know cannot be necessary. My point is that the account of arithmetic presented here does not imply, by itself, that arithmetic is contingent (false of some possible world).

The necessary aspects of a human-created arithmetic indicate that Frege was overly concerned with the possibility of the emergence of arithmetic being contingent on human affairs. Frege argued that, if arithmetic were a human invention,

> astronomers would hesitate to draw any conclusions about the distant past, for fear of being charged with anachronism,—with reckoning twice two as four regardless of the fact that our idea of number is a product of evolution and has a history behind it. It might be doubted whether by that time it had progressed so far. How could they profess to know that the proposition 2×2=4 was already in existence in that remote epoch? Might not the creatures then extant have held the proposition 2×2=5, from which the proposition 2×2=4 was only evolved later through a process of natural selection in the struggle for existence? Why, it might even be that 2×2=4 itself is destined in the same way to develop into 2×2=3! (Frege, 1960, p. xviii-xix).

The fact that astronomers can confidently use a human-created arithmetic when investigating the remote past is easily dealt with by distinguishing between assertions *made in the past* and assertions *made in the present about the past* (analogous to Hale's distinction between *truly asserted in* a possible world and *true of* a possible world which I made use above). One second after the Big Bang, there was no one who could truly assert 2×2=4, since the procedures this statement describes had not been invented yet. However, an astronomer working *today* can truly assert that, one second after the Big Bang, two protons plus two protons made four protons. Astronomers know the counting technology and therefore they can apply it to describe the past, present or future, in the same way that an archaeologist can use radiocarbon dating to find out when a mummy was embalmed in Ancient Egypt (where radiocarbon dating did not exist) and in the same way that we can use arithmetic to speculate about possible worlds. Regarding the "evolutionary scale" from 2×2=5 to 2×2=3, this makes no sense in the light of the normativity of arithmetic as explained above. The correction of 2×2=4 is established by the application of the counting procedure to produce accurate assessments of cardinality. A counting sequence in which, starting from the first position, two leaps of two positions ahead stopped at the fifth or third position would be simply wrong.

Apriority In the above paragraphs we have seen that, since arithmetic is about procedures, it cannot be invalidated by different ways the world could be. This is why arithmetic cannot be refuted by empirical facts. In this sense, arithmetic as construed here is a priori. However, it is not *knowable* a priori. We do need experience by means of exposure and suitable training with a number of counting and calculating practices to acquire knowledge about these practices, as we have seen. The point is that, once we have learned these techniques, it does not matter how things are. It is always possible to segregate space into discrete units, stabilize these units (in reality or in a representation) and apply arithmetical techniques to count and calculate with these units. Humans invented arithmetic to deal with discrete objects, and learn arithmetic by manipulating discrete objects, but the techniques that constitute arithmetic are sufficiently general and flexible to be applied in every environment. It is our autonomy to segregate the world into units and collect these units regardless of the structure of the world that makes arithmetic empirically irrefutable and universally applicable.

Objectivity A few paragraphs above, I argued that a different arithmetic created by cognitive agents not subject to the same cognitive constraints as us should be compatible with ours, since what validates a method of assessing and calculating with cardinalities is correspondence between the outcomes produced by the method and the cardinal size of the collections involved. This suggests that cardinalities, i.e., the cardinal size of the collections to which we apply arithmetical techniques, are the objective subject matter that underlies arithmetic. This is correct, but this point needs further clarification.

Following Frege, I have repeatedly stressed that we are free to segregate the world into units and collect these units as we please. Thus, in a sense, cardinalities are *subjective*: the cardinality of an aggregate of matter is a property assigned to it by a cognitive agent. Cardinalities are *not* mind-independent. This does not mean, however, that there are no objective facts about cardinalities. The point is that the subjective aspect involved in the determination of cardinalities is restricted to the operations of segregation and collection. These are the only factors left to agents' discretion. Once an agent has segregated a sector of the world into discrete units and grouped some of them into a collection, the cardinality of the collection has already been determined. The cardinality of a collection is a procedural consequence of the segregation and grouping acts that gave rise to it. Once cognitive agents have decided upon a sortal, it is no longer up to them to decide how many instances of it there are in a certain sector of the world. A trivial example: it is not up to us to decide whether the solar system has eight or nine planets once the sortal 'planet' has been defined. Given the current definition of 'planet,' a method that counts nine planets instead of eight, i.e., a method that produces a model collection with one item corresponding to each planet plus an extra item, is plainly wrong. We can immediately notice the error by one-to-one correspondence: after each planet is paired with one item of the model collection, one item will remain to which no planet corresponds. In other words, this method would produce a *wrong* model of the cardinal size of the collection of planets.

Ultimately, this is what secures the objectivity of applied numerical statements and, as a result, of pure arithmetical statements (which describe properties of the tools developed to deal with cardinalities accurately). True enough, the expression 'objective knowledge' literally means knowledge of *objects*, but in the current account arithmetic is not concerned with objects. This, however, should not be seen as a problem for my account. In recent decades, there has been a tendency in the philosophy of mathematics to see objectivity as possibly independent of the existence of mathematical objects. The origin of this view is often credited to Kreisel. Dummett (1973, p. 508) succinctly expresses it by saying: "As Kreisel has remarked, what is important is not the existence of mathematical objects, but the objectivity of mathematical statements," which makes room for an explanation of objectivity without recourse to mathematical objects. Tait (2001, p. 22), who attributes the origin of this idea to Cantor, says that "the question of objectivity in mathematics concerns, not primarily the existence of objects, but the objectivity of mathematical discourse."

Although the association between objectivity and objects is no longer automatic in the philosophy of mathematics, philosophers of mathematics have not yet given up on the requirement for mind-independence. Shapiro expresses the dominant view: "Intuitively, to be objective is to be independent of human judgments, conventions, forms of life and the like" (Shapiro, 2011, p. 97). On the current account, arithmetic is obviously mind-dependent. Those who take mind-independence as a necessary condition for objectivity will consider my account misguided from the outset. I have nothing to say to them beyond suggesting that they reconsider their narrow conception of objectivity.

There are two approaches to objectivity discussed in the philosophy of science, both involving mind-dependent aspects, which are helpful for our purposes, namely, *product objectivity* and *process objectivity*. According to the former, "science is objective in that, or to the extent that, its products—theories, laws, experimental results and observations—constitute accurate representations of the external world" (Reiss & Sprenger, 2017, section 1, para. 2). The fact that the external world contains things whose existence is mind-dependent does not matter for this conception of objectivity. Airplanes, brains, human behavior, all of these mind-dependent entities can be the subject matter of an objective investigation; the only requirement for product objectivity is that the final product—e.g., a theory—represents aspects of these entities accurately. The concept of product objectivity allows for an account of the objectivity of the *descriptions* made by number theory and applied arithmetical statements. Number theory is product-objective insofar as it provides correct descriptions of aspects of a certain kind of human behavior, namely, counting and calculation techniques. As discussed in section 7.2, the recitation of the counting sequence and the procedures of moving backwards and forwards on the counting sequence are some of the practices that constitute the objective subject matter that underlies number theory. Cardinalities, in turn, are the objective subject matter that underlies applied numerical statements. The sentence 'The solar system has eight planets' is product-objective insofar as it provides an accurate description of our solar system.

The other conception of objectivity discussed in the philosophy of science that is relevant here, process objectivity, allows for an account of the objectivity of the *prescriptions* made by counting and calculation techniques. According to this notion, "science is objective in that, or to the extent that, the processes and methods that characterize it neither depend on contingent social and ethical values, nor on the individual bias of a scientist" (Reiss & Sprenger, 2017, section1, para. 2). Or, as Douglas (2004, p. 462) puts it, "[t]he key to procedural objectivity is that regardless of who engages in a procedurally objective process, they would do it in the same way, producing the same result." Objective processes in science aim at avoiding distortions in results due to scientists' idiosyncrasies and cognitive biases: "when we call a method objective in the procedural sense, we state that it has been designed in a way that screens out the possibility of individual biases or idiosyncrasies distorting the results" (Koskinen, 2018, p. 10). This is exactly what we have seen with respect to the function of counting and calculating techniques, though at a much more fundamental level. These techniques were designed to screen out individual "biases"—the distortions produced by non-symbolic estimation and calculation. Whereas estimation can vary across individuals, the counting of a given collection, if performed correctly, always produces the same outcome regardless of who is doing the counting. The same applies to other arithmetical techniques (such as calculation algorithms).

Thus, even if arithmetic as construed here is not objective in the narrow sense preferred by some philosophers of mathematics, it is objective in the more plausible senses of objectivity defined by philosophers of science. Furthermore, it is not up to us to choose how objective we want arithmetic to be. In the current account, it is a matter of fact that arithmetic is mind-dependent and not descriptive of a realm of objects. If this is so, then attempts to provide an account of arithmetic that secures its mind-independence are doomed to failure.

The account of arithmetic presented here is in line with platonism in several respects, though with important adjustments at key points. Here, as in platonism, an objective reality underlying number theory does exist. This reality, however, is not a realm of non-spatiotemporal objects, but a toolkit of techniques defined by means of clear-cut, deterministic rules, whose correct functioning is determined by their being proper means to achieve certain ends. Here, as in platonism, number theory can be seen as necessary, although its sentences would not be true *in* all possible worlds, but true *of* all possible worlds. Here, as in platonism, arithmetic is universally applicable. Finally, here, as in platonism, number theory is a priori, though only in the sense that number theory cannot be falsified by evidence coming from perception, not in the sense that it is knowable a priori.

7.4 Comparison with other nominalistic accounts

The idea that arithmetic is about operations, rather than objects, is not new. Wittgenstein and Kitcher already proposed it decades ago. And the idea that arithmetic (and mathematics, more broadly) involves a certain as-if attitude is not new either. Indeed, pretense is the core idea of fictionalism, a trendy approach in contemporary philosophy of mathematics. In this section I briefly point out important differences and some similarities between Wittgenstein's, Kitcher's, and fictionalist accounts of arithmetic when compared to the current account.

Wittgenstein Right in the first section of the *Philosophical Investigations*, Wittgenstein suggests that numerals "encapsulate"—as I could say here, using Sfard's terminology—the operation of counting.

> I send someone shopping. I give him a slip marked "five red apples". He takes the slip to the shopkeeper, who opens the drawer marked "apples"; … then he says the series of

> cardinal numbers—I assume that he knows them by heart—up to the word "five" and for each number he takes an apple ... —It is in this and similar ways that one operates with words (Wittgenstein, 1958, p. 2-3).

Wittgenstein's procedural view on the meaning of numerals has been an important source of inspiration for the present account. This view permeates all of Wittgenstein's writings, both earlier and later, on the philosophy of mathematics (Klenk, 1976; Kremer, 2002). In keeping with Kremer's interpretation of Wittgenstein's account of arithmetical equations in the *Tractatus*, it is possible to draw an even closer parallel with the account I defend here. According to Kremer, Wittgenstein sees an equation such as '15×9=135' as a record of a calculation. The idea is that this expression records the result of performing the multiplication of 15 by 9. We can always calculate the multiplication again if needed, but the equation works as a "shortcut" which dispenses us from having to perform the same process again. In my terminology, this is tantamount to saying that '15×9=135' encapsulates the process of multiplying 15 by 9.

In spite of these remarkable similarities, there are important dissimilarities between my account and Wittgenstein's. One of them involves his claim, in the *Tractatus*, that arithmetical equations are "senseless," or that they "say nothing." It is not trivial to understand what Wittgenstein wants to convey with this claim, but at this point my account and Wittgenstein's clearly diverge. We have seen that, in the current account, arithmetical equations are not only meaningful, but true (if correct, of course). According to Kremer, for Wittgenstein, arithmetical equations "say nothing" because what they would say cannot be said, only shown. Equations supposedly *show* "how one is to go on" in inferences involving numbers (Kremer, 2002, p. 298), but they cannot *say* how to go on because processes cannot be expressed by propositions (Kremer, 2002, p. 299).

But, be it Kremer's interpretation only or Wittgenstein's real intention, this claim does not seem plausible. We describe procedures all the time, for example, by giving a sequence of steps. True enough, descriptions of complex procedures can easily become cumbersome (this is where recourse to reification comes in). It is also true that descriptions of procedures sometimes fall short of providing all that is needed for their reproduction. This is especially true of counting and arithmetic in general, where purely verbal descriptions are of no help to those who have not yet mastered the relevant techniques. If this was what Wittgenstein had in mind, he was right to point out this limitation. But what is helpful in such cases is not showing a formula, but showing the execution of the technique itself. Recall that children do not learn to count and calculate solely by following verbal instructions or seeing formulas, but mainly by seeing someone counting and calculating and then being actively guided in their attempts to do the same. It is not that arithmetic equations "say nothing," but that they do not say enough about the processes they encapsulate. They say only what is essential for those who already know how to reproduce the processes they encapsulate.

Another way of understanding Wittgenstein's claim that arithmetical equations are senseless is to see them as similar to tautologies, in that they could not divide the space of possible facts because they supposedly show the very logic of the world. "The logic of the world which the propositions of logic show in tautologies, mathematics shows in equations" (Wittgenstein, 2001, §6.22, p. 79). This is another enigmatic claim, but again it is clear that it is another point where my account and Wittgenstein's come apart. As we have seen, arithmetic is universally applicable, regardless of how the world is. Since arithmetic is compatible with a variety of possible worlds, it is unlikely that it will show "the logic of the world" (of which world?). Perhaps all possible worlds, even the ones most different from ours, share a set of basic structural features, which would be shown by tautologies and arithmetical equations. But, again, this does not seem plausible in light of the current account. As I claimed above, we are free to segment any environment into units and collect them regardless of the structure of the world, and this is what makes arithmetical statements insensitive to empirical refutation. Thus, it is more likely that arithmetic describes "the logic" of certain human activities we engage in in this world, which can be applicable to other possible worlds, rather than the logic of any world.

Kitcher In Kitcher's account, as in the current account, arithmetic is true in virtue of certain operations. Besides this similarity, however, there are many differences. Here, I highlight three of them. The first regards the operations that are thought to underlie arithmetic. For Kitcher, these operations are those of collecting (or merging) and segregating.

> I begin with an elementary phenomenon. A young child is shuffling blocks on the floor. A group of his blocks is segregated and inspected, and then merged with a previously scrutinized group of three blocks … Children come to learn the meanings of 'set', 'number', 'addition' and to accept basic truths of arithmetic by engaging in activities of collecting and segregating (Kitcher, 1984, p. 107-108).

As we have seen, this scenario is not empirically plausible. Children do not start learning numbers by collecting and segregating objects, but by learning by heart a sequence of initially (for them) meaningless words, as we saw in Chapter 4. Only much later, when the child has already mastered counting, will she be able to appreciate that, e.g., two groups of three blocks merged with each other make a group of six blocks. This is not to deny that the operations of collection and segregation are relevant to arithmetic. As noted above, the cardinality of a collection is a consequence of our actions of segregating matter into units and collecting these units. However, these operations by themselves cannot trigger the formation of number concepts and therefore do not underlie arithmetic. Only tallying and counting practices can do that: these are the operations that underlie arithmetic.

The second relevant difference is that, for Kitcher, the operations that underlie arithmetic are not performed by ordinary humans, but by an ideal agent. This is how Kitcher responds to the problem of providing an infinite ontology for arithmetic so as to make statements about infinity true. In his account, the bound variables of arithmetical statements range over operations. Of course, there are not enough operations in the world to fulfill the universe of arithmetic, and thus his only alternative is to postulate ideal operations performed by an ideal agent. The ontological status of this agent and her operations, though, are anything but clear (Hoffman, 2004). Recall that my account does not face this problem because, here, the variables of arithmetical statements range over discursive objects (i.e., reifications).

The third point of contention I want to highlight is Kitcher's view on the relationship between arithmetic and the world. He claims that arithmetic is true not only in virtue of the ideal operations performed by the ideal agent, but also in virtue of the structure of the world. The idea is that operations such as segregation and collection reveal "the mathematical structure of reality" (Kitcher, 1984, p. 107).

> [T]o present my thesis in a way which will bring out its realist character, we might consider arithmetic to be true in virtue not of what *we can do* to the world but rather of what *the world* will let us do *to* it. To coin a Millian phrase, arithmetic is about 'permanent possibilities of manipulation.' More straightforwardly, arithmetic describes those structural features of the world in virtue of which we are able to segregate and recombine objects: the operations of segregation and recombination bring about the manifestation of underlying dispositional traits (Kitcher, 1984, p. 108).

The claim is that, because the world "affords" segregation and collection, we can know that the structure of the world is such that it "affords" these operations. As he puts it, "mathematics is an idealized science of particular universal affordances" (Kitcher, 1984, p. 12). But we have already seen that every structure can be viewed as affording (or at least allowing for) segregation, collection and counting for those who already know how to perform these operations. True enough, judging by all the evidence we have, it seems that our world affords quite natural ways of segregating and collecting objects. This fact about our world may be behind the evolutionary pressures that gave us quantical cognition and that made the invention of tallying and counting advantageous. But although these structural properties of the world may have awakened us to the development of arithmetic, the cognitive tools that compose arithmetic do not depend on the existence of such affordances to be usable. We are able to segregate, collect and count even when these structural features are not present. Whether or not segregation and collection carves reality at its joints is completely irrelevant to the truth of applied and pure arithmetical statements.

Fictionalism In the current account, numbers do not exist but we speak of them as if they do. Hence, discourse about numbers as construed here can be seen as involving a certain degree of pretense. After all, when we speak of numbers as if they are objects, we are making claims that are false if literally interpreted. Is arithmetic a kind of fictional story? Fictionalists think so (though for reasons which have nothing to do with reification). According to Balaguer (2018, para. 2), fictionalism "is the view that (a) our mathematical sentences and theories do purport to be about abstract mathematical objects, as platonism suggests, but (b) there are no such things as abstract objects, and so (c) our mathematical theories are not true." Clearly, my approach here does not qualify as fictionalist because, although I subscribe to (a) and (b), I take arithmetical statements to be true. This is a consequence of the triadic semantics developed in section 7.1, which allows for a reconciliation between the referentiality of arithmetical statements and the non-existence of the objects they purport to denote that does not compromise their truth, as we saw in section 7.2.

Balaguer calls those who accept (a) and (b) but reject (c) "deflationary-truth nominalists," a category in which he places Azzouni (2010), whose notion of truth-value inducers I adopted here. According to Balaguer, deflationary-truth nominalists advance an alternative conception of truth. Whereas fictionalists adopt a homogeneous semantic theory in which truth conditions for non-mathematical and mathematical statements are the same, deflationary-truth nominalists adopt a dual theory of truth, in which the truth-conditions for mathematical statements do not follow the same rules as those for other parts of language. According to Balaguer, this is what Azzouni does by proposing that the truth-value inducers of sentences wherein empty names occur are not the putative objects they fail to refer to, which is at odds with the truth-conditions for sentences where only genuine names occur. Then, according to Balaguer,

> the central claim behind that view [deflationary-truth nominalism] is an empirical hypothesis about ordinary discourse. In particular, it's a claim about the meaning of the term 'true,' or about the concept of truth. When deflationary-truth nominalists say that, e.g., '3 is prime' could be true even if there were no such thing as the number 3, they are making a claim about the ordinary concept of truth. They are saying that that concept applies in certain situations that most of us—platonists and fictionalists and just about everyone else—think it *doesn't* apply in …
>
> Given this, most fictionalists would probably say that the problem with deflationary-truth nominalism is that it's empirically implausible. In other words, the objection would be that deflationary-truth nominalism flies badly in the face of our intuitions about the meaning of 'true.' And there does seem to be some justification for this claim. For instance, it just seems intuitively obvious that the sentence 'Mars is a planet' could not be literally true unless there really existed such a thing as Mars. Moreover, intuitively, the sentence 'Mars is a planet, but it doesn't exist' seems like a contradiction, and this intuition seems to be incompatible with deflationary-truth nominalism (Balaguer, 2018, section 1.3, para. 2 and 3).

I agree with Balaguer that the point of contention here is empirical. However, the empirical facts to which an adequate conception of truth must respond are not our intuitions about the concept of truth, as he claims, but our *use* of the concept of truth.[14] Although our intuitions at first glance do say that, if we are referring to certain objects, our statements are to be judged true or false with regard to those objects, this is not how we effectively evaluate truth in practice. Figurative speech is a remarkable example of this. The sentence "It's raining cats and dogs" is judged to be asserting a truth if it is raining very hard, but its truth has nothing to do with cats and dogs. Another example is the sentence "The average family in the UK has 1.9 children," discussed above. Does the fictionalist postulate the existence of the average familiy in the UK or does she take this sentence to be false? Viewing the average family in the UK as a reification seems more plausible. The evaluation of the truth of metaphorical and reificatory sentences does not comply with what Balaguer takes our intuitions about truth to be. There is empirical evidence for discourse about numbers being a kind of reified discourse, as we saw in previous chapters. This is manifested in the process through which we acquire number concepts and develop our ideas and discourse on numbers. If talk of numbers as objects is really reificatory, it is a mistake to evaluate the truth value of numerical statements as if they were literal. Fictionalists make this mistake.

[14] This is not only a Wittgensteinian observation, but also a standard assumption in the empirical investigation of concepts. "Concepts are typically opaque: People do not have a privileged access to the content of their concepts, and a thinker is not able to articulate the content of a concept just because she possesses it. As a result, people can have mistaken views about their own concepts, and, if asked to spell out their content (that is, to explain how they think about something), they may formulate a mistaken theory. The right way to articulate the content of a concept is to ask people to use it (e.g., to apply it) and to infer its content from its use" (Machery, 2017, p. 210).

This brings to the fore the hermeneutic character of the approach I am proposing here. We saw in Chapter 1 that Burgess distinguishes two classes of nominalisms/fictionalisms: hermeneutic and revolutionary. Hermeneutic approaches offer interpretations of mathematical language and practice which are intended to reveal what, "contrary to superficial appearances, deep-down mathematical language has meant all along" (Burgess, 2004, p. 23). This is what I am doing here with respect to arithmetic: I am claiming that number talk was never about genuine objects, but about discursive objects that encapsulate certain procedures. Revolutionary nominalists/fictionalists, by contrast, "concede that their reconstructions of mathematics are not analyses of current mathematics, but amendments to it; not exegeses, but emendations" (Burgess, 2004, p. 23), and then advocate for the replacement of current mathematical practices and ways of speaking by new ones. We saw in Chapter 1 that fictionalists have troubles with Burgess's dichotomy. For example, Leng agrees with Burgess that hermeneutic fictionalism is not viable: "Burgess rightly rejects hermeneutic fictionalism on the grounds that it is not supported by the evidence of mathematical practice" (Leng, 2005, p. 277). Balaguer concurs: "hermeneutic fictionalism is implausible and unmotivated; as an empirical hypothesis about what mathematicians intend, there is simply no good evidence for it, and it seems obviously false" (Balaguer, 2018, section 2.3, para. 2). But the revolutionary approach is not an option for fictionalists either: "given the comparative historical records of success and failure of philosophy on the one hand, and of mathematics on the other, to propose philosophical 'corrections' to mathematics is *comically immodest*" (Burgess, 2004, p. 30).

Whereas fictionalism is admittedly empirically implausible on the one hand, and risks "comical immodesty," on the other, an account where reification occupies the role of pretense at the origins of the idea that numbers exist is empirically plausible and, at the same time, explains why mathematical practice should not change. Pretense requires overt prescriptions to imagine, a make-believe attitude that is unlikely to go unnoticed. Reification, by contrast, is usually a non-intentional process which takes place by a kind of "contamination" of discourse about processes by ways of speaking typical of discourse about objects, and we do have evidence for this in arithmetic, as we saw in Chapter 6. This "contamination" is cognitively beneficial, does not imply falsity, and is not to be avoided, but encouraged: it gives us numbers, powerful cognitive tools.

7.5 An empirically refutable argument

The above comparison between my account and other nominalistic accounts highlighted some important differences. But the fundamental difference between the current account and others offered in the philosophy of mathematics is the fact that it is empirically informed and, hence, empirically refutable. In this section, I briefly summarize the argument presented in this dissertation for the non-existence of numbers. I also offer some examples of the kind of empirical evidence that might refute it.

My argument for the non-existence of numbers can be divided into two parts: the first, presented in Chapters 2 to 6, concerns a scientific investigation of the origins of our beliefs and concepts regarding numbers; the second, presented in this chapter, gives a philosophical account of the semantic and epistemic attributes of arithmetic in light of the findings presented in the first part. The following is a summary of the main conclusions of the first part:

(a) When we examine the ontogenetic and historical roots of our numerical beliefs, we find out that they do not come from a realm of objects, but from experience with certain procedures involving external symbolic resources, initially experienced as meaningless tokens.

(b) These procedures are aimed at enabling us to determine the cardinal size of collections of discrete items accurately; familiarity with these procedures gives us number concepts, which endow those initially meaningless tokens with meaning.

(c) The idea that numbers are objects is due to a process of reification of these techniques which takes place when children start learning to calculate. As higher-order counting operations, calculations become easier when lower-order operations are reified.

This is what a scientific investigation of the phenomenon tells us. In sum, the idea is that numbers are as-if objects originated in a context wherein numbers conceived of as genuine objects played no role. This reverses the platonist schema according to which first there were the eternal numbers, then we somehow gained knowledge of them and thus became able to count and calculate. The sequence of events described in (a)-(c) reveals that it is actually the other way around: first there were certain symbolic practices created and used by people who did not know numbers, then number concepts (certain mental contents), and finally numbers (discursive objects). Those eternal numbers of the traditional schema apparently do not exist.

Nevertheless, this does not conclusively show that numbers as genuine objects do not exist unless we are able to explain how arithmetic can display the semantic and epistemic attributes we usually ascribe to it—such as referentiality, truth, and objectivity—in the absence of existing numbers. After all, in the philosophy of mathematics, numbers have been postulated precisely to account for these semantic and epistemic features. My goal in the previous sections of this chapter was to show that focusing on the techniques and operations responsible for the emergence of our ideas about numbers suffices to explain the semantic and epistemic attributes of arithmetic. The main conclusions of this second part of the argument are the following:

(d) In a triadic semantics, the referentiality of numerical statements belongs to the intermediate mental level: numerals refer to discursive/intentional objects, although they do not denote anything in the external world.

(e) Unsuccessful denotation does not make numerical statements false, since they are true of the procedures that the discursive objects they refer to encapsulate.

(f) These procedures are governed by clear-cut deterministic rules and have properties of their own. Number theory is objective because it describes structural properties of these processes.

Other properties of arithmetic, such as normativity, necessity, and apriority are dealt with as discussed in section 7.3. Once the postulation of extant numbers has been shown to be dispensable to account for the semantic and epistemic features of arithmetic, we are in a position to conclude that numbers do not exist by making use of Ockham's razor. However, it is worth noticing that here, the dispensability of numbers is not just a consequence of a clever logical reconstrual of arithmetic, but an observational matter. It was an empirical investigation into the phenomenon of arithmetic as embedded in human practices and human cognition that revealed that the explanatory function traditionally ascribed to numbers as existing objects is in fact fulfilled by procedures encapsulated within numerical statements. Extant numbers are made superfluous by the current account in almost the same way that phlogiston was made superfluous by the chemistry of the late eighteenth century. The only difference is that, whereas phlogiston was thought to be a spatiotemporal substance, numbers are claimed to be outside of space and time. This is the only reason why Ockham's razor must be evoked here, since a direct proof of the non-existence of non-spatiotemporal numbers is impossible.

As I have repeatedly pointed out, this argument for the non-existence of numbers does not refute the hypothesis that non-spatiotemporal numbers exist, because this hypothesis is irrefutable. The dispensability of extant numbers does not necessarily imply their nonexistence. However, the irrefutability of extant numbers does not recommend belief in them either, nor does it recommend agnosticism—that is, unless one is also happy to accept the existence of Russell's teapot revolving around the Sun, as argued in section 6.5.

Surely, the nominalistic account of arithmetic presented here is susceptible to all sorts of philosophical counter-arguments. But it is particularly susceptible to empirically informed counter-arguments. As new findings in numerical cognition, developmental psychology, history, and anthropology shed more light on the historical and ontogenetic origins of our numerical ideas, the account presented here might become progressively outdated, requiring corrections or major revisions (that is, if new counter-evidence emerges). The reliance of the current account on non-nativist accounts of the ontogenesis of numerical cognition is a particularly sensitive point. At present, the non-nativist hypothesis, according to which our innate quantical skills are non-numerical, seems to be best supported by currently available scientific evidence, as I argued in Chapter 3. However, nativist accounts are still alive and kicking, and it is possible that new scientific findings will tilt the balance towards them. In this case, if it is shown that number concepts are already in place at birth, the idea that our numerical beliefs result from the internalization of cognitive tools invented by human beings, as I have claimed, can no longer be tenable. If number concepts are shown to be innate, counting and calculation techniques are more likely to be *externalizations* of these inborn skills. Consequently, my explanation of the truth of arithmetical statements in virtue of the counting procedure would be, at best, incomplete. We would have to investigate the phylogenetic origins of inborn number concepts in order to find out what they reflect (if anything). If it is shown that they reflect properties of the environment, then there might be a sense in which one could argue that numbers exist in the physical world, thus vindicating the Millian idea according to which arithmetic reflects the most general structural properties of the world.

It is difficult to conceive of how a scientific/empirical investigation of the phenomenon of numerical cognition could offer support to the platonist hypothesis, due to its speculative nature. However, original Platonism could benefit from a turn of events in favor of nativism, since the existence of innate number concepts is in line with Plato's hypothesis of reminiscence. A kind of Kantianism could also benefit from nativism, insofar as inborn number concepts could be conceived of as framing our perception of the environment, rather than reflecting it.

As for the role of reification in the idea that numbers are objects, new findings about how children progress in arithmetic at school can challenge the hypothesis that they start with a predominantly procedural view which is progressively reified as they start learning more complex arithmetical operations. Fictionalists such as Leng (2010), in particular, might be interested in this stage of development, to investigate whether pretense plays some role in mathematical learning. Prescriptions to imagine are not completely absent from mathematics classes. For example, think of a teacher asking pupils to imagine a line that never ends in order to explain the number line. In schools, fictionalists may be able to find a way of providing empirical plausibility for their claims. (That is, if philosophers become convinced of the importance of empirical evidence to the philosophy of mathematics.)

7.6 Conclusion

In this chapter, I have provided an account of the semantic and epistemic characteristics usually ascribed to arithmetic as a function of the procedures that underlie numerical cognition. I started this chapter by showing that it is possible to view numerals as referring terms in the absence of existing numbers, providing we adopt a more realistic semantic approach in which the role of human minds is not downplayed. In this approach, numerals refer to discursive/intentional objects. This semantic account set the stage for the explanation of the truth and objectivity of numerical statements in virtue of the processes the discursive objects they refer to encapsulate. I also showed that the contingency of arithmetic in human affairs does not preclude its universal applicability. In a sense, arithmetic can be viewed as necessary: at first glance, it may be true of all possible worlds, since there is no way a world could be that would prevent the execution of the processes underlying its statements or invalidate these processes as accurate techniques to deal with discrete quantities. This also shows that arithmetic is a priori, i.e., not subject to refutation by means of empirical observation. Arithmetic does not describe the "mathematical structure of the world" as some have argued, but structural properties of counting and calculation procedures. Finally, I showed that the current account, in contrast to other more "traditional" philosophical accounts of arithmetic, is accountable to empirical data, and can thus be improved or refuted on an empirical basis.

Concluding remarks

This dissertation presented an empirically informed nominalistic account of the nature of the positive integers. After a brief discussion of the shortcomings of a priori accounts of the ontological status of numbers, I started with a investigation into the way we acquire numerical competence and develop the idea that numbers are objects based on empirical research done on these issues. This investigation then informed a philosophical approach to the metaphysics of arithmetic in which existing numbers are replaced by procedures (the counting procedure and higher-order procedures performed over the counting procedure) as the objective subject matter that grounds the truth and objectivity of arithmetical statements. The *idea* that numbers are objects, though, still has a role to play: numbers conceived of as as-if objects are essential for simplifying calculation, as discussed in section 6.3. In this sense, numbers are cognitive tools.

Although appeal to beneficial consequences is not a proper way to defend an empirical hypothesis, it must be noted that the account presented here has many advantages over other accounts of the nature of numbers available in the literature. It is epistemically less problematic than platonism (since here, the objective subject matter that arithmetic describes is accessible to us) and fictionalism (since here, arithmetical statements can be true). In the present account, arithmetical knowledge is not only possible but also simply explainable by means of a standard causal theory of knowledge. The account presented here is also superior to some forms of nominalism in that it does not demand modifications to the way we read and interpret arithmetical statements. This explains why there is no contradiction implicit in the subtitle of this dissertation. We can speak of the *nature* of non-existing numbers because numbers are discursive objects, i.e., they appear in language as if they are real objects, but in fact they do not exist as such: what exists are the relevant procedures. Here, the cultural nature of arithmetic is key, but arithmetic is not reduced to social conventions or collective agreements. Finally, these theoretical benefits are achieved by mobilizing familiar concepts and practices most of us have experienced as children. Highly idealized solutions that are logically impeccable but unrealistic (such as recourse to an ideal agent (Kitcher, 1984)) play no role here.

An important point which I did not address in this dissertation concerns the possibility of extending this approach to other branches of mathematics. After all, I have argued only that the positive integers do not exist, and from this it does not follow that other mathematical entities, such as sets, real numbers, and geometric forms do not exist. This should not be seen as a shortcoming. In the spirit of the current account, we should not expect that an account of the nature of the positive integers could provide an across-the-board explanation of the nature of all mathematical entities. Since the account of the nature of the positive integers presented here was based on empirical findings (rather than universal principles), the extension of this approach to other categories of numbers and branches of mathematics would require empirical evidence of the sort reviewed in chapters 2-6 to lead to conclusions about their existence.

Thus, if the current account were to be extended to cover, for example, the negative integers, we would need scientifically gathered data showing that the very use of symbols for negative integers in certain routines and mathematical techniques is what gives us the idea that negative integers exist. Indeed, in this case the extension seems to be simple. A negative number can be seen as encapsulating the outcome of the subtraction of n from m where n and m are natural numbers and $n>m$. This is empirically plausible in light of empirical results showing that negative numbers are encoded in the mind as natural numbers tagged with a separate polarity sign (Gilmore et al., 2018, section 7.4). Rational numbers, in turn, can be seen as encapsulating the outcome of a division of integers (Sfard, 1991). Sfard (1991) also suggests that a similar approach can be applied to geometric forms. For example, a circle may reify the operation of rotating a compass around a fixed point. But certainly, more data is needed to substantiate these conclusions.

The amount of data available in the field of mathematical cognition about higher branches of mathematics is still small in comparison to data about arithmetic. However, it is possible to anticipate that the current account is likely to face major difficulties with regard to non-constructible mathematical objects. Constructible objects are easy to deal with within this account, since the symbolic procedures performed to construct them are the natural candidates providing the corresponding concepts and the targets of reification by encapsulation. But objects whose existence is proved only non-constructively do not have a known process of construction. Then what are they? It seems obvious that non-constructible mathematical objects cannot result from reifications produced by encapsulation. A possibility to be explored is whether other processes of reification have a role to play with respect to non-constructible objects. Another possibility is that non-constructibe objects require a completely different explanation. Perhaps we will ultimately conclude that non-constructible objects are fictions, or postulations, or even platonic objects. My point is that this is not something that we can conclude aprioristically.

Philosophers like universal explanations that can account for a large class of phenomena in a uniform manner. But sometimes (or often!) reality is more diverse than what we would like and we have to live with this. We cannot approach reality in its full complexity from our armchairs equipped only with general principles, intuitions and plausible reflections. We have rejected such an attitude since the so-called “scientific revolution” when it comes to the investigation of the physical world. The general message of this dissertation is that this also applies to the investigation of what has been called the metaphysics of mathematics.

Glossary

alienation The act of suppressing from discourse the agents who reify or perform the reified procedures. See section 6.1.

ANS Approximate Number System. The innate cognitive system thought to be responsible for the implementation of the capacity to estimate. See section 3.2.

CP-knower A child who can understand the cardinality principle and, as a result, can pass the Give-a-Number test for all the number words in the initial segment of the counting sequence she already knows. See section 4.3.

de-semantification The act of temporally "turning off" the semantical content of symbols when they are mechanically manipulated in accordance with operational rules aimed at fulfilling a cognitive task. See section 2.4.

discursive object A reification that occupies the usual place of an object in speaking and thinking. See section 6.1.

encapsulation The process of reification in which a noun is assigned to processes, operations or actions by means of which narratives about these non-objects can be presented as stories about objects. See section 6.1.

enculturation The thesis according to which higher cognitive capacities result from transformations in the brain driven by the cultural environment. See section 2.3.

estimation The ability to produce an approximate appraisal of the number of items in a collection, without active verbal counting. Estimation error grows as quantity grows; it is subject to the effects of Weber's law and sensory adaptation. See section 3.1.1.

number The referent of a numeral.

number concept A concept, here, is understood as a mental content. Accordingly, number concepts are the mental contents that endow numerals with meaning and underlie numerical competence. See section 4.1.

numeral Any culturally created symbol that refers to a number. Examples: 'eight', '8', 'VIII'.

numerical cognition Is both the name of the set of abilities numerate humans possess to deal with numbers and the discipline that investigates how human beings and other animals deal with information we usually see as numerical.

numerosity a property of perceptual interactions between an agent and a stimulus, evaluated by means of quantical skills, that refers to the perceived cardinal magnitude of the stimulus as a function of the sortal used by the agent's perceptual system to identify and collect discrete items in the stimulus. See section 3.5.3.

OFS Object File System. The innate cognitive system thought to be responsible for the implementation of the capacity to subitize. See section 3.2.

platonism The view according to which abstract entities (objects, structures, propositions, etc.) exist outside of space and time. This term is spelled with lower-case initial so as to differentiate it from Platonism, the doctrine held by Plato, since contemporary platonists are unlikely to endorse all of Plato's views on this matter.

quantical cognition The set of non-symbolic abilities shared by humans and some non-human animals to perceive numerosities by means of subitizing and estimation. Also known as *non-symbolic numerical cognition*. See Chapter 3.

re-semantification The process through which the de-semantification of a symbolic system during its mechanical operation makes room for the creation of a new, original interpretation of the symbols. See section 2.4.

reification The act of conceiving of a non-object as if it were an object. Reification takes place when ways of thinking and speaking characteristic of discourse about physical objects are transplanted to discourse about non-objects. See section 6.1.

subitizing The fast and accurate enumeration of collections up to three or four elements without active verbal counting. See section 3.1.1.

subset-knower A child who knows the meaning of the words for numbers smaller than three or four only, and does not understand the cardinality principle yet. See section 4.3.

working realism The practice of speaking of numbers and doing arithmetic as if numbers exist. It is the resulting position towards the existence of numbers after the process of reification of number concepts has taken place. See section 6.2.

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Summary

In everyday parlance, we rarely distinguish between numbers and numerals. But this distinction is necessary in mathematical, linguistic and philosophical contexts. Numerals are symbols for numbers; numbers are the *meanings* of numerals. This is what allows us to say that 'ten,' '10,' and 'X' all have *the same* meaning.

Radical skeptics aside, no one doubts that numerals exist. I can show some numerals here: '3,' 'eleven,' 'X' (these very marks on the screen or on paper are numerals). However, the situation is much more controversial when it comes to numbers. Even if I tried to show a number by writing down 3 or by uttering /three/, the only thing that I would have shown or uttered would be, again, a numeral. If I said "look, 3 is not this symbol, but the object it refers to," a natural reply would be "but where is this object?" To this question, I couldn't answer "Three is here [pointing to a collection with three elements]," since, if three is there, then three cannot be also here: ◊◊◊, which is a different collection. Perhaps I could say that three is *instantiated* by collections of three elements, but this implies that three itself is something else.

This and other puzzles have led philosophers to assume that, if numbers exist at all, they must be outside of space and time. This assumption, however, also leads to problems. If numbers are outside of space and time, they are completely out of our reach, and then it becomes a mystery how we can acquire numerical knowledge. On the other hand, if we simply deny that numbers exist, we will face difficulties in explaining how arithmetical statements such as "2 is even" can be true. If two does not exist, it cannot be anything, let alone even.

Various candidate solutions have been proposed to these problems. I review some of them in Chapter 1. A problem common to all the reviewed proposals is that there is no way of deciding which one is the correct one based on independent evidence. This is the case (or so I claim) because all these accounts disregard empirical evidence that is crucial to decide between them, on the basis of the assumption that, if numbers exist, they must be outside of space and time.

In this dissertation, I reject this assumption and take empirical data about how we acquire numerical knowledge as the starting point. Research in the fields of numerical cognition, developmental psychology and mathematics education has progressed fast in the last decades and has revealed important aspects of the processes through which we acquire knowledge of numbers at a young age.

The picture that emerges from these findings (reviewed in Chapters 2-6) can be summarized by paraphrasing a famous biblical passage (John 1:1): "In the beginning was the Word, and the Word was used in mechanical procedures, and through them all numbers

were made." The 'Words' here are numerals, naturally. The idea is that numerals and the procedures wherein they are used, such as counting and calculation, are the elements from which numbers are "made up." Let me unpack this.

Research in developmental psychology has shown that children start learning the sequence of counting words as meaningless sounds recited during what they experience as a mechanical procedure, where each act of pointing to an object is accompanied by the utterance of a word. In the beginning, children do not know what the words mean nor the purpose of this procedure. Over time, they start learning the meaning of number words progressively. They do not learn their meanings by associating each word with a specific object or class of objects (as they probably learn the meaning of 'apple,' for example), but by using these words in a procedure that, as they understand later, aims at determining the cardinal size of collections accurately. It is the use of initially meaningless numerals in the counting procedure that generates the numerals' own meanings, as discussed in Chapter 4.

Historically, it is also plausible that the first "counting" techniques used words that originally did not have numerical meanings. In many small-scale cultures around the world, from New Guinea to the Amazon Forest, body-part tallies fulfill the same function that counting does in our culture. In body-part tallies, the names of body parts or descriptions of the gestures made replace our counting words. If modern counting systems originated from ancient body-part tallying systems, then it is plausible that originally non-numerical words (body part names or descriptions of gestures) gave rise to the first numerical concepts. Again, if this was so, it was the use of words initially seen as non-numerical in a procedure resembling the counting procedure (tallying) that generated the words' own numerical meanings, as discussed in Chapter 5.

Now we can return to the distinction between numerals and numbers. The observation that both in history and individual development number words produce their own meanings suggests that numbers are not objects whose existence is independent of the words that designate them. Rather, the *idea* that there is a number corresponding to each numeral is produced by the very use of numerals. It is not like seeing an apple and learning that the word 'apple' designates that kind of fruit. We never see a number; we learn certain words that are recited in a stable order during a certain procedure. Later we realize the function of these words, and then understand their meanings. The object designated by 'nine' is nowhere, not because it is outside of space and time, but because it does not exist. The *idea* that it exists is a product of the use of this symbol for calculation, as discussed in Chapter 6.

In this account, however, the nonexistence of numbers does not imply that numerical statements are false. Once we have established that there are symbolic procedures behind what we call numbers, we can see that what we say about numbers is, in fact, about those procedures. A simple example illustrates this point. That 2+2 is equal to 4 is not a truth about objects called 2 and 4; rather, it is a truth about an operation: "count 'one,' 'two,' and then count 'one,' 'two' again, and then count what you have counted in the previous steps together; this final counting will stop at 'four'." In other words, 2+2=4 describes a fact about the counting procedure, rather than a fact about numbers conceived of as existing objects (this is developed in Chapter 7). We speak of numbers *as if* they were objects because it is simpler and cognitively useful. The idea that numbers are objects makes calculation easier. This is why numbers, these "objects" that exist only in discourse, are cognitive tools.

Curriculum Vitae

César Frederico dos Santos (Araranguá, Santa Catarina, Brazil, 1980) obtained bachelor's degrees in Information Systems (2004) and Philosophy (2010) at the Federal University of Santa Catarina. He obtained a master's degree in Philosophy (2012) at the same university, under the supervision of prof.dr. Antonio Mariano Nogueira Coelho, writing his thesis on naturalistic approaches to the philosophy of set theory. In 2013, he became an assistant professor at the Department of Philosophy of the Federal University of Maranhão (UFMA), in São Luís, Brazil. In 2017, the UFMA granted him a paid leave to complete his studies at the University of Groningen, under the supervision of prof.dr. Catarina Dutilh Novaes. In 2018, following his supervisor, he moved to the Vrije Universiteit Amsterdam, where this dissertation was finished.